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Option Pricing, Arbitrage and Martingales[1]

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This paper provides a short introduction into the mathematical foundations of the theory of valuing derivative securities. We discuss the mathematical setting of option price theory and derive the relationship between no-arbitrage and martingales. We provide examples how this theory can be applied to equity and foreign-exchange derivatives. We also explain how the theory can be applied to interest rate derivative securities and provide some examples.

1. Introduction

Options are financial instruments that entitle the owner to buy other financial instruments (the *underlying instruments*) at a fixed date (the *maturity date*) at a fixed price (the *exercise price* or *strike*). The owner does not have the obligation to exercise the option. Hence, he will only do so if the market price of the underlying instrument is above the exercise price. These options are more specifically named (European-style) *call options*. Also *put options* are traded which entitle the holder to sell the underlying instrument at the maturity date against the exercise price.

The person that has to sell the instruments if a call option is exercised is called the *writer* of the call option. The same terminology is used for put

[1] This paper is based on Chapters 2, 3 and 4 of A. Pelsser [15], *Efficient Methods for Valuing and Managing Interest Rate and Other Derivative Securities*, Ph.D. Thesis, Erasmus University Rotterdam.

options. The writer will only give the option to the holder if the latter pays a certain price, the *option premium.* This price, of course, depends on today's price of the underlying instrument, the exercise price and the maturity date.

The cornerstone of option pricing theory is the assumption that any financial instrument which has a non-negative payout must have a non-negative price. The existence of an instrument which would have non-negative payoffs and a negative price is called an *arbitrage opportunity.* If arbitrage opportunities would exist, it would be a means for investors to generate money without any initial investment. Of course, many investors would try to exploit the arbitrage opportunity, and due to the increased demand, the price would rise and the arbitrage opportunity would disappear. Hence, in an economy that is in equilibrium it seems reasonable to rule out the existence of arbitrage opportunities. Although the assumption that arbitrage opportunities do not exist seems a rather plausible and trivial assumption, we shall see it is indeed the foundation for all of the option pricing theory.

Another important assumption needed is the absence of transaction costs. This means that assets can be bought and sold in the market for the same price. This assumption is clearly violated in real markets. In the presence of transaction costs, not all arbitrage opportunities which would theoretically be profitable can be exploited. However, large market participants (like banks and institutions) face very little transaction costs. These large players have the opportunity to exploit almost all arbitrage opportunities with large amounts of money and markets will be driven to an equilibrium close to the equilibrium that would prevail if transaction costs were absent. Hence, if we consider markets as a whole, the assumption that transaction costs are absent is a good approximation of the real world situation. Individual investors often face larger transaction costs when trading in financial markets, and they often use options and other derivative securities to enlarge their opportunity set.

The rest of this paper is organised as follows. In the next section we discuss the mathematical setting of option price theory and derive the relationship between no-arbitrage and martingales. We illustrate the concepts with the well-known Black-Scholes model and with a model for pricing foreign currency options. Section 3 is devoted to interest rate derivative securities, where we explain the Heath-Jarrow-Morton model and the forward-risk-adjusted measure. We use the Ho-Lee model as an illustration.

2. Arbitrage and Martingales

In this section we provide the basic mathematical setting in which the theory of option pricing can be cast. We derive the key result that an economy is free of arbitrage opportunities if a probability measure can be found such that the prices of marketed assets become martingales. By setting up trading strategies which replicate the payoff of derivative securities, the martingale property can then be shown to carry over from the marketed assets to the prices of all derivative securities. Hence, the prices of all derivatives become martingales and this property can then be used to calculate prices for derivative securities.

2.1. Basic Setup

Throughout this paper we consider a continuous trading economy, with trading interval given by $[0, T]$. The uncertainty is modeled by the probability space $(\Omega, \mathcal{F}, Q)$. In this notation, Ω denotes a sample space, with elements $\omega \in \Omega$; $\mathcal{F}$ denotes a σ-algebra on Ω; and Q denotes a probability measure on $(\Omega, \mathcal{F})$. The uncertainty is resolved over $[0, T]$ according to a filtration $\{\mathcal{F}_t\}$ satisfying "the usual conditions", generated by a Brownian Motion W initiated at 0. We also assume that there exist assets which are traded in a market. The assets are called *marketed assets.* We also assume that the prices $Z(t)$ of these marketed assets can be modeled via Itô processes which are described by stochastic differential equations

$$dZ(t) = \mu(t, \omega)\, dt + \sigma(t, \omega)\, dW, \tag{1}$$

where the functions $\mu(t, \omega)$ and $\sigma(t, \omega)$ are assumed to be $\mathcal{F}_t$-adapted and also satisfy

$$\begin{aligned} \textstyle\int_0^T |\mu(t, \omega)|\, dt &< \infty \\ \textstyle\int_0^T \sigma(t, \omega)^2\, dt &< \infty, \end{aligned} \tag{2}$$

with probability one. The observant reader may note that there is only one source of uncertainty (the Brownian Motion W) that drives the prices of the marketed assets. It is relatively straightforward to set up the economy such that more sources of uncertainty drive the prices of the marketed assets. However, one source of uncertainty is enough for a large part of option pricing theory. For example, in case of options on a stock, the economy is not the economy of all traded financial instruments, but restricted to the underlying stock, a riskless bond and all (call- and put-)options with different maturities and exercise prices on this stock. Hence, we only assume that the option prices are influenced by movements in the price of the underlying and non-stochastic variables such as the time to maturity and the exercise price. It is also true that the prices of marketed assets defined in (1) are less general than usual in the literature. The sample paths of Itô processes are continuous, which excludes discrete dividend payments. It is possible to develop a general theory on the basis of right-continuous with left-limits (RCLL) processes[2] which allow for a countable number of discontinuities in the sample paths. However, in this paper we will nowhere encounter marketed assets with discontinuous sample paths. Suppose there are N marketed assets with prices $Z_1(t), \ldots, Z_N(t)$, which all follow Itô processes. A *trading strategy* is a predictable N-dimensional stochastic process $\delta(t, \omega) = \big(\delta_1(t, \omega), \ldots, \delta_N(t, \omega)\big)$, where $\delta_n(t, \omega)$ denotes the holdings in asset n at time t. The asset holdings $\delta_n(t, \omega)$ are furthermore assumed to satisfy an additional regularity condition to which we will return later. The value $V(\delta, t)$ at time t of a trading strategy δ is given by

[2] RCLL processes are also known as *càdlàg*, *corlol* or *R-processes* in the literature.

$$V(\delta, t) = \sum_{n=1}^{N} \delta_n(t) Z_n(t). \tag{3}$$

A *self-financing trading strategy* is a strategy δ with the property

$$V(\delta, t) = V(\delta, 0) + \sum_{n=1}^{N} \int_0^t \delta_n(s)\, dZ_n(s), \quad \forall t \in [0, T]. \tag{4}$$

Hence, a self-financing trading strategy is a trading strategy that requires nor generates funds between time 0 and time T. An *arbitrage opportunity* is a self-financing trading strategy δ, with $\Pr[V(\delta, T) \geq 0] = 1$ and $V(\delta, 0) < 0$. Hence, an arbitrage opportunity is a self-financing trading strategy which has strictly negative initial costs, and with probability one has a non-negative value at time T.

A *derivative security* is defined as a $\mathcal{F}_T$-measurable random variable $H(T)$. The random variable has to satisfy an additional regularity constraint to which we will return later. The random variable $H(T)$ can be interpreted as the (uncertain) payoff of the derivative security at time T. If we can find a self-financing trading strategy δ such that $V(\delta, T) = H(T)$ with probability one, the derivative is said to be *attainable*. The self-financing trading strategy is then called a *replicating strategy*. If in an economy all derivative securities are attainable, the economy is called *complete*. If no arbitrage opportunities and no transaction costs exist in an economy, the value of a replicating strategy at time t gives a unique value for the attainable derivative $H(T)$. This is true, since (in the absence of transaction costs) the existence of two replicating strategies of the same derivative with different values would immediately create an arbitrage opportunity. Hence, we can determine the value of derivative securities by the value of the replicating portfolios. This is called *pricing by arbitrage*. However, this raises two questions. First, under which conditions is a continuous trading economy free of arbitrage opportunities? Second, under which conditions is the economy complete? If these two conditions are satisfied, all derivative securities can be priced by arbitrage.

2.2. Equivalent Martingale Measure

The questions of no-arbitrage and completeness were analysed in the seminal papers of HARRISON AND KREPS [8] and HARRISON AND PLISKA [9]. They showed that both questions can be solved at once, if we express prices in the economy in terms of a numeraire. Any marketed asset which has strictly positive prices (and pays no dividends) for all $t \in [0, T]$ is called a *numeraire*. We can use numeraires to denominate all prices in an economy. Suppose that the marketed asset Z_1 is a numeraire. The prices of other marketed assets denominated in Z_1 are called the *relative prices* denoted by $Z'_n = Z_n/Z_1$. Let $(\Omega, \mathcal{F}, Q)$ denote the probability space from the previous subsection. Consider now the set $\mathcal{Q}$ which contains all probability measures Q^* such that:

i) Q^* is equivalent to Q, i.e. both measures have the same null-sets;

ii) the relative price processes Z'_n are martingales under Q^* for all n, i.e. for $t \leq s$ we have $E^*(Z'_n(s) \mid \mathcal{F}_t) = Z'_n(t)$.

The measures $Q^* \in \mathcal{Q}$ are called *equivalent martingale measures.* Suppose we take one equivalent martingale measure $Q^* \in \mathcal{Q}$. Then, in terms of this "reference measure", we can give precise definitions for derivative securities and trading strategies given in the previous subsection. A *derivative security* is a $\mathcal{F}_T$-measurable random variable $H(T)$ such that the expectation $E^{Q^*}\left(|H(T)|\right) < \infty$ holds.[3] Hence, derivative securities are those securities for which the expectation of the payoff is well-defined. A *trading strategy* is a predictable N-dimensional stochastic process $\left(\delta_1(t,\omega), \ldots, \delta_N(t,\omega)\right)$ such that the stochastic integrals

$$\int_0^t \delta_n(s)\, dZ'_n(s) \tag{5}$$

are martingales under Q^*. For self-financing strategies this implies that the value $V'(\delta, t)$ in terms of the relative prices Z' is a Q^*-martingale. The condition on trading strategies is a rather technical condition. It arises from the fact that for predictable processes in general, the value processes $V'(\delta, t)$ of self-financing trading strategies are only *local* martingales under Q^*. For a local martingale

$$\sup_{t\in[0,T]} \left\{E^{Q^*}\left(V'(\delta,t)\right)\right\} = \infty \tag{6}$$

is possible, while for martingales

$$\sup_{t\in[0,T]} \left\{E^{Q^*}\left(V'(\delta,t)\right)\right\} < \infty \tag{7}$$

is always satisfied. This difference between local martingales and martingales allows for the existence of so-called *doubling strategies*, which are arbitrage opportunities. This was first pointed out by HARRISON AND PLISKA [9]. For an example of a doubling strategy in the economy we have set up here, see DUFFIE [5]. Hence, an economy can only be arbitrage-free if the value processes of self-financing trading strategies are martingales. Several restrictions can be imposed on the processes δ to ensure the martingale property of the value processes $V'(\delta, t)$. For a discussion, see HARRISON AND PLISKA [9] or DUFFIE [4]. HEATH and JARROW [10], DOTHAN [3] and DUFFIE [5] show that the presence of wealth constraints or constraints like margin requirements also ensures that the value processes are martingales. Because these constraints are actually present in security markets, it will be assumed throughout this paper that this restriction holds. Subject to the definitions given above, we have the following result:

[3] The set of all random variables that satisfy these constraints is often denoted $\mathcal{L}^1(\Omega, \mathcal{F}_T, Q^*)$.

Unique Equivalent Martingale Measure. *A continuous economy is free of arbitrage opportunities and every derivative security is attainable if $\mathcal{Q}$ contains exactly one equivalent martingale measure.*

For numeraires which are non-stochastic this result was proved by Harrison and Pliska [9]. For a general proof, see Dothan [3], Chapter 12. □

The definition of $\mathcal{Q}$ depends on the choice of numeraire. Hence, for a different choice of numeraire we obtain a different set $\mathcal{Q}$ of equivalent martingale measures and a different unique equivalent martingale measure. Hence, "unique" only means that the set $\mathcal{Q}$ contains only one element. To make things worse, the definition of derivative securities changes also with a different choice of numeraire. It is therefore conceivable that a payoff pattern $H(T)$ which can be replicated for one choice of numeraire, cannot be replicated for another choice of numeraire. Much works remains to be done in this area, for example in establishing which set of payoff patterns can be replicated for all choices of numeraire.

From the result given above follows immediately that for a given numeraire Z with unique equivalent martingale measure Q_Z, the value of a self-financing trading strategy $V'(\delta,t) = V(\delta,t)/Z(t)$ is a Q_Z-martingale. Hence, for a replicating strategy δ_H that replicates the derivative security $H(T)$ we obtain

$$E^{Q_Z}\Big(\frac{H(T)}{Z(T)} \;\Big|\; \mathcal{F}_t\Big) = E^{Q_Z}\Big(\frac{V(\delta_H,T)}{Z(T)} \;\Big|\; \mathcal{F}_t\Big) = \frac{V(\delta_H,t)}{Z(t)}, \tag{8}$$

where the last equality follows from the definition of a martingale. Combining the first and last expression yields

$$V(\delta_H,t) = Z(t)E^{Q_Z}\Big(\frac{H(T)}{Z(T)} \;\Big|\; \mathcal{F}_t\Big). \tag{9}$$

This formula can be used to determine the value at time $t < T$ for any derivative security $H(T)$. The theorem of the Unique Equivalent Martingale Measure was first proved by Harrison and Kreps [8]. In their paper they used the value of a riskless money-market account as the numeraire. Later it was recognised that the choice of numeraire is arbitrary. However, for this historic reason, the unique equivalent martingale measure obtained by taking the value of a money-market account as a numeraire, is called "the" equivalent martingale measure, which is a very unfortunate name. In this paper we will stick to this convention.

To illustrate the concepts developed here, we will apply them to the well known Black-Scholes [2] framework. However, before we do so, we give two results which can be used for explicit calculations in the case of Brownian Motions.

2.3. Girsanov's Theorem and Itô's Lemma

A key result which can be used to explicitly determine equivalent martingale measures in the case of Brownian Motions is *Girsanov's Theorem.* This theorem provides us with a tool to change the drift of a Brownian Motion by changing the probability measure.

GIRSANOV'S THEOREM. *For any stochastic process $\kappa(t)$ such that*

$$\int_0^t \kappa(s)^2\, ds < \infty,$$

with probability one, consider the Radon-Nikodym derivative $\rho(t)$ given by

$$\rho(t) = \exp\Big\{\int_0^t \kappa(s)\, dW(s) - \tfrac{1}{2}\int_0^t \kappa(s)^2\, ds\Big\}$$

then under the measure $dQ^* = \rho\, dQ$ the process

$$W^*(t) = W(t) - \int_0^t \kappa(s)\, ds$$

is also a Brownian Motion. DOTHAN [3] provides several generalisations and proofs of this theorem. From the last equation in Girsanov's Theorem follows that

$$dW = dW^* + \kappa(t)\, dt \tag{10}$$

which is a result we will often use.

Another key result from stochastic calculus is known as *Itô's Lemma.* Given a stochastic process x described by a stochastic differential equation, Itô's Lemma allows us to describe the behaviour of stochastic processes derived as functions $f(t,x)$ of the process x.

ITÔ'S LEMMA. *Suppose we have a stochastic process x given by the stochastic differential equation $dx = \mu(t,\omega)\, dt + \sigma(t,\omega)\, dW$ and a function $f(t,x)$ of the process x, then f satisfies*

$$df = \left(\frac{\partial f(t,x)}{\partial t} + \mu(t,\omega)\frac{\partial f(t,x)}{\partial x} + \right.$$
$$\left. \tfrac{1}{2}\sigma(t,\omega)^2 \frac{\partial^2 f(t,x)}{\partial x^2}\right) dt + \sigma(t,\omega)\tfrac{\partial f(t,x)}{\partial x}\, dW,$$

provided that f is sufficiently differentiable.

For a proof of Itô's Lemma, see ARNOLD [1], Sections 5.3 and 5.4. □

2.4. Application: Black-Scholes Model

Let us now consider the BLACK AND SCHOLES [2] option pricing model. Using this setting enables us to illustrate the concepts developed. In the Black-Scholes economy there are two marketed assets: B which is the value of a riskless money-market account with $B(0) = 1$ and a stock S. The prices of the assets are described by the following (stochastic) differential equations

$$\begin{aligned} dB &= rB\,dt \\ dS &= \mu S\,dt + \sigma S\,dW. \end{aligned} \tag{11}$$

The money-market account is assumed to earn a constant interest rate r, and the stock price is assumed to follow a geometric Brownian Motion with constant drift μ and constant volatility σ. The value of the money-market account is strictly positive and can serve as a numeraire. Hence, we obtain the relative price $S'(t) = S(t)/B(t)$. From Itô's Lemma we obtain that the relative price process follows

$$dS' = (\mu - r)S'\,dt + \sigma S'\,dW. \tag{12}$$

To identify equivalent martingale measures we can apply Girsanov's Theorem. For $\kappa(t) \equiv -(\mu - r)/\sigma$ we obtain the new measure $dQ^* = \rho\,dQ$ where the process S' follows

$$\begin{aligned} dS' &= (\mu - r)S'\,dt + \sigma S'\left(dW^* - \tfrac{\mu - r}{\sigma}\,dt\right) \\ &= \sigma S'\,dW^* \end{aligned} \tag{13}$$

which is a martingale. For $\sigma \neq 0$ this is the only measure which turns the relative prices into martingales, and the measure Q^* is unique.[4] Therefore, the Black-Scholes economy is arbitrage-free and complete for $\sigma \neq 0$. Under the measure Q^*, the original price process S follows the process

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\left(dW^* - \tfrac{\mu - r}{\sigma}\,dt\right) \\ &= rS\,dt + \sigma S\,dW^*. \end{aligned} \tag{14}$$

We see that under the equivalent martingale measure the drift μ of the process S is replaced by the interest rate r. The solution to this stochastic differential equation can be expressed as

$$S(t) = S(0)\exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma W^*(t)\}, \tag{15}$$

where $W^*(t)$ is the value of the Brownian Motion at time t under the equivalent martingale measure. We now extend the economy with options that have this particular stock as underlying instrument. We already discussed the difference between call and put options. A further distinction between different options can be made. Most options can be exercised at any time *before* the maturity date. However, as we explained before, some options can only be exercised at maturity. This latter class is called *European*, while the first class is called *American*. Hence, the distinction has nothing to do with the physical place where the options are traded. A European call option with strike K has at the maturity date T a value of $H(T) = \max\{S(T) - K, 0\}$. If the introduction of options into the economy should not give rise to arbitrage opportunities

[4] This is, of course, just a heuristic argument. For rigorous proofs, the interested reader is referred to the Harrison-Kreps and Harrison-Pliska papers.

it follows from (9) that the price of the option $\mathbf{C}(0)$ at time 0 is given by $E^*\big(\max\{S(T)-K,0\}/B(T)\big)$. To evaluate this expectation, we use the explicit solution of $S(T)$ under the equivalent martingale measure given in (15) and we get

$$\begin{aligned}&E^*\big(\max\{S(T)-K,0\}/B(T)\big) = \\ &\int_{-\infty}^{\infty} e^{-rT}\max\{S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma w}-K,0\}\frac{e^{-\frac{1}{2}\frac{w^2}{T}}}{\sqrt{2\pi T}}\,dw.\end{aligned} \tag{16}$$

A straightforward calculation will confirm that this integral can be expressed in terms of cumulative normal distribution functions $N(.)$ as follows

$$\mathbf{C}(0) = S(0)N(d) - e^{-rT}KN(d-\sigma\sqrt{T}) \tag{17}$$

with

$$d = \frac{\log\left(\frac{S(0)}{K}\right) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{18}$$

which is the celebrated Black-Scholes option pricing formula. The pricing of American options can also be cast in the framework developed here. However, it is a problem which is considerably more complicated than the pricing of European options. Since American options give the owner of the option the right to exercise at any time before maturity, an optimal stopping problem has to solved to determine the value of an American option. The interested reader is referred to DUFFIE [5] and the references mentioned therein for an introduction.

In the derivation given above, we used the value of a money-market account B as a numeraire. However, this choice is arbitrary. The stock price S is also strictly positive for all t and can also be used as a numeraire. If we choose S as a numeraire, we obtain from Itô's Lemma that the relative price $B' = B/S$ follows

$$dB' = (r-\mu+\sigma^2)B'\,dt - \sigma B'\,dW. \tag{19}$$

If we apply Girsanov's Theorem with $\kappa = (r-\mu)/\sigma + \sigma$, we obtain (for $\sigma \neq 0$) the unique equivalent martingale measure Q_S for which the relative price B' is a martingale. From (9) we obtain

$$\begin{aligned}\mathbf{C}(0) &= S(0)E^{Q_S}\left(\frac{\max\{S(T)-K,0\}}{S(T)}\right) \\ &= S(0)E^{Q_S}\big(\max\{1-K\frac{1}{S(T)},0\}\big).\end{aligned} \tag{20}$$

Using Itô's Lemma and Girsanov's Theorem we obtain for the equivalent martingale measure Q_S, that the process $1/S$ follows

$$\begin{aligned}d\frac{1}{S} &= (-\mu+\sigma^2)\frac{1}{S}\,dt - \sigma\frac{1}{S}\big(dW^S + (\frac{r-\mu}{\sigma}+\sigma)dt\big) \\ &= -r\frac{1}{S}\,dt - \sigma\frac{1}{S}\,dW^S,\end{aligned} \tag{21}$$

where W^S is a Brownian Motion under Q_S. The explicit solution can be expressed as

$$\frac{1}{S(t)} = \frac{1}{S(0)} \exp\{(-r - \tfrac{1}{2}\sigma^2)t - \sigma W^S(t)\}. \tag{22}$$

Using this explicit form, we can evaluate the expectation (20). It is left to the reader to verify that this also gives the Black-Scholes formula (17).

2.5. Application: Foreign-Exchange Options

The example for the Black-Scholes economy given above of the influence of the numeraire is a bit artificial, however a more fruitful application can be found when we consider foreign-exchange (F/X) options. These are options which have a foreign currency as the underlying instrument. The first valuation formula for F/X-options in a Black-Scholes setting was given by GARMAN AND KOHLHAGEN [6]. This formula is nowadays widely used by F/X-option traders all over the world. An interesting aspect of F/X-derivatives is that we can either calculate the value of a derivative in the domestic market or in the foreign market. For example, a call option to buy dollars for D-marks is equivalent to a put option to sell D-marks for dollars. If the economy is arbitrage-free, both values must be the same, otherwise an "international" arbitrage opportunity would arise. Consider the following, very simple, international economy. In the domestic market D there is a money-market account B^D, which earns an instantaneous riskless interest rate r^D; in the foreign country F there is also a money-market account B^F with interest rate r^F. Let X denote the exchange rate. This is the value in domestic currency of one unit of foreign currency. The exchange rate follows a geometric Brownian Motion. The three price processes can be summarised as

$$\begin{cases} dB^F &= r^F B^F \, dt, \\ dX &= \mu X \, dt + \sigma X \, dW, \\ dB^D &= r^D B^D \, dt. \end{cases} \tag{23}$$

From a domestic point of view, there are two basic marketed assets: the domestic money-market account B^D and the value of the foreign money-market account in domestic terms, given by $B^F X$. From Itô's Lemma we obtain that the process $(B^F X)$ follows

$$d(B^F X) = (r^F + \mu)(B^F X) \, dt + \sigma(B^F X) \, dW. \tag{24}$$

The domestic money-market account can be used as a numeraire, and the relative price process $(B^F X)' = (B^F X)/B^D$ follows the process

$$d(B^F X)' = (r^F - r^D + \mu)(B^F X)' \, dt + \sigma(B^F X)' \, dW. \tag{25}$$

An application of Girsanov's Theorem with $\kappa(t) \equiv -(r^F - r^D + \mu)/\sigma$ will yield the domestic unique equivalent martingale measure Q^D under which the relative price process $(B^F X)'$ is a martingale. Under the domestic measure Q^D, the exchange rate process follows

$$dX = (r^D - r^F)X\,dt + \sigma X\,dW^D, \tag{26}$$

which is the process used in the Garman-Kohlhagen formula. We can also take the perspective of the foreign market. Here we also have two marketed assets: B^F and (B^D/X). Using B^F as a numeraire, we obtain the relative price process $(B^D/X)' = (B^D/X)/B^F$ which follows the process

$$d\left(\frac{B^D}{X}\right)' = (r^D - r^F - \mu + \sigma^2)\left(\frac{B^D}{X}\right)' dt - \sigma\left(\frac{B^D}{X}\right)' dW. \tag{27}$$

If we apply Girsanov's Theorem with $\kappa(t) \equiv (r^D - r^F - \mu)/\sigma + \sigma$, we obtain the foreign unique equivalent martingale measure Q^F. Under the foreign measure Q^F, the foreign exchange rate $1/X$ follows the process

$$d\left(\frac{1}{X}\right) = (r^F - r^D)\left(\frac{1}{X}\right) dt - \sigma\left(\frac{1}{X}\right) dW^F. \tag{28}$$

This process is exactly the right process for calculating the Garman-Kohlhagen formula in the foreign market. Hence, in this economy a trader in the domestic market and a trader in the foreign market will calculate exactly the same price for a F/X-option. For more examples of calculating prices of derivatives under domestic and foreign martingale measures, see Reiner [16]. He uses these measures repeatedly to calculate the value of so-called *quanto options*, which are options on foreign assets denominated in the domestic currency.

3. Pricing Interest Rate Derivatives

In the previous sections, we summarised the theory behind the valuation of derivative securities in general. However, in the examples we have given, we made the assumption that it is possible to trade in the underlying values. In the Black-Scholes economy, the underlying value is the asset price in which the agents in the economy can trade. When we want to consider the valuation of interest rate derivatives, the picture is slightly different. The underlying value of interest rate derivatives is the *spot interest rate*. The spot interest rate is the instantaneous riskless return earned by putting your money for a very short instant dt in a money-market account. All other interest rates are derived from the spot interest rate. These rates are just different ways of quoting the current prices of bonds and swaps traded in the market. The prices of these instruments are determined by the spot interest rate. However, we cannot trade in the spot interest rate itself (nor in any other interest rate), it is only possible to trade in interest rate derivatives. In this section we show how prices of interest rate derivatives can be calculated using equivalent martingale measures, which is the methodology of HEATH, JARROW AND MORTON [11]. We will only derive the equivalent martingale measure for one-factor interest rate models, this will allow us to explain the essence of the Heath-Jarrow-Morton (HJM) methodology.

3.1. Forward Rates

A *discount bond* is an instrument which pays 1 at maturity T in all states of the world. The price of a discount bond with maturity T at time $t < T$ is denoted by $P(t,T)$. The marketed assets which can be traded (like bonds and swaps) can all be expressed in terms of discount bonds with different maturities. For an introduction, see HULL [13]. In their set-up HJM choose not to model discount bond prices directly, but to model the prices of *forward rates* $f(t,T)$. The forward rate is given by

$$f(t,T) = \frac{-\partial \log P(t,T)}{\partial T}; \tag{29}$$

it is the instantaneous interest rate one can contract for at time t to invest in the money-market account at time T. It is easy to see that the spot interest rate $r(t)$ is equal to $f(t,t)$. HJM assume that the forward rates satisfy the following equation

$$f(t,T) - f(0,T) = \int_0^t \alpha(s,T,\omega)\, ds + \int_0^t \sigma(s,T,\omega)\, dW(s), \tag{30}$$

where ω denotes the state of the world. Equation (30) is the integral form of the stochastic differential equation

$$df(t,T) = \alpha(t,T,\omega)\, dt + \sigma(t,T,\omega)\, dW. \tag{31}$$

The stochastic process for the forward rates defined above is very general. The functions α and σ are allowed to depend on the maturity T of the forward rate and are allowed to depend on the state of the world ω. The spot interest rate $r(t)$ is equal to $f(t,t)$, hence we get from (30)

$$r(t) = f(0,t) + \int_0^t \alpha(s,t,\omega)\, ds + \int_0^t \sigma(s,t,\omega)\, dW(s). \tag{32}$$

Using (29) we can express the discount bond prices in terms of the forward rates as

$$\log P(t,T) = -\int_t^T f(t,s)\, ds. \tag{33}$$

Substituting (30) into this equation and by interchanging the order of integration and simplifying, HJM obtain the following process for the discount bond prices (suppressing the notational dependence on ω)

$$dP(t,T) = b(t,T)P(t,T)\, dt + a(t,T)P(t,T)\, dW, \tag{34}$$

where

$$\begin{aligned} a(t,T,\omega) &= -\int_t^T \sigma(t,s,\omega)\,ds \\ b(t,T,\omega) &= r(t) - \int_t^T \alpha(t,s,\omega)\,ds + \tfrac{1}{2}a(t,T,\omega)^2. \end{aligned} \tag{35}$$

3.2. Equivalent Martingale Measure

Having specified the stochastic process followed by the discount bonds $P(t,T)$ which are the marketed assets, we want to establish the existence of an equivalent martingale measure to ensure that no arbitrage opportunities can exist in the economy. Suppose we keep reinvesting money in the money-market account. Every instant dt the money market account earns the riskless spot interest rate and the value $B(t)$ of the money-market account is given by $dB = rBdt$. If we solve this ordinary differential equation we get

$$B(t) = \exp\Big\{\int_0^t r(s)\,ds\Big\}. \tag{36}$$

As in the Black-Scholes economy of Section 2.4, the value of the money-market account is strictly positive and can be used as a numeraire. Hence, in the HJM economy we obtain the relative prices $P'(t,T) = P(t,T)/B(t)$. Itô's Lemma yields

$$dP'(t,T) = \big(b(t,T) - r(t)\big)P'(t,T)\,dt + a(t,T)P'(t,T)\,dW. \tag{37}$$

The HJM economy will be arbitrage-free if we can find a unique equivalent probability measure such that the relative prices P' of the discount bonds become martingales. Suppose we consider the discount bond with maturity T_1. If we apply Girsanov's Theorem with $\kappa(t,T_1) = -\big(b(t,T_1) - r(t)\big)/a(t,T_1)$, we obtain under the new measure $dQ^*_{T_1} = \rho(t,T_1)\,dQ$ that the process $P'(t,T_1)$ is a martingale. This change of measure depends on the maturity of the discount bond T_1 and will only make this particular discount bond a martingale. However, we want to find an equivalent martingale measure that changes all marketed assets, that is all discount bonds, to martingales. This is only possible if the ratio $\big(b(t,T,\omega) - r(t)\big)/a(t,T,\omega)$ is independent of T. Let $\lambda(t,\omega)$ denote the common value of this ratio, if we apply then Girsanov's Theorem with $\kappa(t,\omega) = -\lambda(t,\omega)$ we get that all discount bonds $P'(t,T)$ are martingales under the equivalent martingale measure $dQ^* = \rho\,dQ$. Since the prices of all discount bonds are dependent on the spot interest rate r, the drift term $b(t,T,\omega)$ cannot be specified arbitrarily. A unique equivalent martingale measure can only be found if the drift term is of the form

$$b(t,T,\omega) - r(t) = \lambda(t,\omega)a(t,T,\omega). \tag{38}$$

The function $\lambda(t,\omega)$ is known as the *market price of risk* at time t in state ω. Substituting the definitions for a and b given in (35), and differentiating with respect to T we find that the drift terms of the forward rate processes $\alpha(t,T,\omega)$ are restricted to

$$\alpha(t,T,\omega) = \sigma(t,T,\omega)\Big(\int_t^T \sigma(t,s,\omega)\,ds + \lambda(t,\omega)\Big). \tag{39}$$

3.3. Calculating Prices

Now that we have determined under which conditions an equivalent martingale measure exists in the HJM model, we can calculate the prices of interest rate derivatives. In Section 2 we derived the result that under the equivalent martingale measure the relative prices $V(t,r)/B(t)$ are martingales. In the HJM economy we get that the price of a financial instrument with a payoff $H(T,r)$ at time T is given by

$$V(t,r) = E^*\Big(e^{-\int_t^T r(s)\,ds} H(T,r) \,\Big|\, \mathcal{F}_t\Big) \tag{40}$$

where the expectation E^* is taken with respect to the equivalent martingale measure Q^*. From Girsanov's Theorem and using the restriction on α given in (39) we obtain that under the equivalent martingale measure the process r follows

$$r(t) = f(0,t) + \int_0^t \sigma(s,t,\omega) \int_s^t \sigma(s,u,\omega)\,du\,ds + \int_0^t \sigma(s,t,\omega)\,dW^*(s). \tag{41}$$

It is clear that for a given initial term-structure of interest rates and for a given choice of the function $\sigma(t,T,\omega)$ the spot rate process under the equivalent martingale measure is completely determined.

3.4. Example: Ho-Lee Model

To illustrate the HJM methodology, we turn to the continuous-time limit of the Ho and Lee [12] model, which we also call the Ho-Lee model. In the Ho-Lee model it is assumed that the function $\sigma(t,T,\omega)$ is constant, and that the market price of risk is a function $\lambda(t)$ of time only. We obtain from (39) that the drift terms of the forward rates are restricted to

$$\alpha(t,T) = \sigma\big(\sigma(T-t) + \lambda(t)\big). \tag{42}$$

Hence, under the equivalent martingale measure, the spot interest rate follows the process

$$r(t) = f(0,t) + \tfrac{1}{2}\sigma^2 t^2 + \sigma W^*(t). \tag{43}$$

If we use the definition of the forward rates given in (29), this equation can also be written in differential form as

$$dr = \theta^*(t)dt + \sigma\,dW^*, \tag{44}$$

where

$$\theta^*(t) = -\frac{\partial^2}{\partial t^2}\log P(0,t) + \sigma^2 t. \tag{45}$$

Using (40) the price of a discount bond can be expressed as

$$P(t,T) = E^*\Big(e^{-\int_t^T r(s)\,ds} \cdot 1 \,\Big|\, \mathcal{F}_t\Big). \tag{46}$$

Hence, the price of a discount bond is given by

$$P(t,T) = E^* \left(e^{-y(T)} \right), \tag{47}$$

where the random variable y is defined as

$$y(T) = \int_t^T r(s)\, ds. \tag{48}$$

Substituting the solution of the stochastic differential equation (44) for r into the definition of $y(t)$ yields

$$y(T) = \int_t^T r(t)\, ds + \int_t^T \int_t^s \theta^*(u)\, du\, ds + \int_t^T \int_t^s \sigma\, dW^*(u)\, ds. \tag{49}$$

By interchanging the order of integration[5] and simplifying we obtain

$$y(T) = r(t)(T-t) + \int_t^T \theta^*(s)(T-s)\, ds + \int_t^T \sigma(T-s)\, dW^*(s). \tag{50}$$

Hence, the process $y(T)$ has a normal distribution with mean

$$m(T) = r(t)(T-t) + \int_t^T \theta^*(s)(T-s)\, ds \tag{51}$$

and variance

$$v(T) = \int_t^T \sigma^2 (T-s)^2\, ds = \tfrac{1}{3}\sigma^2 (T-t)^3. \tag{52}$$

From this follows that the expectation (47) can be evaluated as

$$\begin{aligned} P(t,T) &= \exp\{-m(T) + \tfrac{1}{2} v(T)\} \\ &= \exp\Big\{ -r(t)(T-t) - \textstyle\int_t^T \theta^*(s)(T-s)\, ds + \tfrac{1}{6}\sigma^2 (T-t)^3 \Big\}. \end{aligned} \tag{53}$$

Hence, the forward rate $f(t,T)$ can be expressed, using (29) as

$$f(t,T) = r(t) + \int_t^T \theta^*(s)\, ds - \tfrac{1}{2}\sigma^2 (T-t)^2. \tag{54}$$

[5] For a proof of Fubini's Theorem for stochastic integrals, see the Appendix of Heath, Jarrow and Morton [11].

3.5. Forward-Risk-Adjusted Measure

In Section 3.3 we derived the result that prices of interest rate derivatives can be calculated by taking the expectation with respect to the equivalent martingale measure. The price $V(t,r)$ of an interest rate derivative can be expressed using (40). In general, this expectation can be too complicated to evaluate, since the two terms of the product inside the expectation both depend on the spot interest rate r. Hence, both terms are correlated for general payoffs. When we determined the equivalent martingale measure in the HJM economy, we used the value of the money-market account $B(t)$ as a numeraire. However, this is not the only choice we could have made. As was argued in Section 2, it is possible to use any financial instrument with a strictly positive price (and no intermediate payouts) as a numeraire. The following theorem shows that if an equivalent martingale measure can be found with respect to one numeraire, then an equivalent martingale measure can be found with respect to any numeraire.

CHANGE OF NUMERAIRE THEOREM. *Let Q^* be the equivalent martingale measure with respect to a numeraire $B(t)$. Let $X(t)$ be an arbitrary numeraire. Then there exists a probability measure Q_X such that:*

(i) for any asset $S(t)$ we have $S(t)/X(t)$ is a martingale with respect to Q_X;

(ii) the Radon-Nikodym derivative that changes the equivalent martingale measure Q^* into Q_X is given by

$$\frac{dQ_X}{dQ^*} = \frac{X(T)}{X(0)B(T)}.$$

A proof can be found in GEMAN et al. (1995). □

The Change of Numeraire Theorem is very powerful in the context of valuing interest rate derivatives. Instead of using the value of the money-market account $B(t)$ as a numeraire, the prices of discount bonds $P(t,T)$ can also be used as a numeraire. A very convenient choice is to use the discount bond with maturity T as a numeraire for derivatives which have a payoff $H(T, r(T))$ at time T. If we denote the probability measure associated to the numeraire $P(t,T)$ by Q_T we can apply the Change of Numeraire Theorem as follows. Under the measure Q_T the prices $V(t,r)/P(t,T)$ are martingales for $t < T$. Hence, applying the definition of a martingale we obtain

$$E^{Q_T}\big(V(T,r(T))/P(T,T) \bigm| \mathcal{F}_t\big) = V(t,r)/P(t,T). \tag{55}$$

However, at time T the price of the discount bond $P(T,T) \equiv 1$ and the price of the instrument V is given by its payoff $V(T,r(T)) = H(T,r(T))$. So, the equation reduces to

$$V(t,r) = P(t,T)E^{Q_T}\big(H(T,r(T)) \bigm| \mathcal{F}_t\big). \tag{56}$$

If we compare this expression with (40), we see that we have managed to express the expectation of the discounted payoff as a discounted expectation of the payoff. We have eliminated the problem of the correlation between the discounting term and the payoff term. The measure Q_T has another very interesting property, which actually gave it the name T-forward-risk-adjusted measure.[6] Under the T-forward-risk-adjusted measure, the forward rate $f(t,T)$ is equal to the expected value of the spot interest rate at time T. The following argument shows why this is true. A discount bond $P(t,T)$ has a payoff of 1 at time T. Using (40), the price of the discount bond can be expressed as

$$P(t,T) = E^*\Big(e^{-\int_t^T r(s)\,ds} \cdot 1 \;\Big|\; \mathcal{F}_t\Big). \tag{57}$$

Taking derivatives with respect to T yields

$$\begin{aligned} -\tfrac{\partial}{\partial T}P(t,T) &= E^*\Big(e^{-\int_t^T r(s)\,ds} r(T) \;\Big|\; \mathcal{F}_t\Big) \\ &= P(t,T)E^{Q_T}\big(r(T) \;\big|\; \mathcal{F}_t\big), \end{aligned} \tag{58}$$

where we have used (56) in the last step. Using the definition of the forward rates $f(t,T) = -\partial/\partial T \log P(t,T)$ we can simplify this expression to

$$f(t,T) = E^{Q_T}\big(r(T) \;\big|\; \mathcal{F}_t\big), \tag{59}$$

which is the desired result.

3.6. Example: Ho-Lee Model (revisited)

In this section we provide an example to illustrate the concepts developed. We consider again the continuous-time limit of the Ho-Lee model we have already encountered in Section 3.4. For this model we can explicitly determine the T-forward-risk-adjusted measure from the Radon-Nikodym derivative In Section 3.4 we showed that, if we use the value of the money-market account $B(t)$ as a numeraire, under the equivalent martingale measure Q^* the spot interest rate r follows the process

$$dr = \theta^*(t)\,dt + \sigma\,dW^*, \tag{60}$$

where $\theta^*(t)$ is given by

$$\theta^*(t) = -\frac{\partial^2}{\partial t^2}\log P(0,t) + \sigma^2 t. \tag{61}$$

The prices at time $t = 0$ of interest rate derivatives with payoff $H(r(T))$ at maturity T can be calculated as

$$V_0 = E^*\left(e^{-\int_0^T r(s)\,ds} H(r(T))\right). \tag{62}$$

[6] This name was coined by JAMSHIDIAN [14], who also provided the argument we use here.

Instead of using the value of the money-market account $B(t)$, we can also use the value of the T-maturity discount bond $P(t,T)$ as a numeraire. We explained that prices can also be calculated as

$$V_0 = P(0,T)E^{Q_T}\big(H(r(T))\big). \tag{63}$$

The Change of Numeraire Theorem provides us with the appropriate Radon-Nikodym derivative to change from Q^* to Q_T

$$\frac{dQ_T}{dQ^*} = \frac{P(T,T)}{P(0,T)B(T)} = \frac{e^{-\int_0^T r(s)\,ds}}{P(0,T)} \tag{64}$$

If we substitute the solution of the stochastic differential equation (60) into this equation and interchange the order of integration we obtain for dQ_T/dQ^*

$$\exp\Big\{ \begin{array}{l} -\log P(0,T) - \int_0^T \Big(-\frac{\partial^2 \log P(0,s)}{\partial s^2} + \sigma^2 s\Big)(T-s)\,ds \\ -\int_0^T \sigma(T-s)\,dW^*(s) \end{array} \Big\}. \tag{65}$$

If we work out the integrals, we see that the Radon-Nikodym derivative can be simplified to

$$\frac{dQ_T}{dQ^*} = \exp\Big\{-\tfrac{1}{6}\sigma^2 T^3 - \int_0^T \sigma(T-s)\,dW^*(s)\Big\}. \tag{66}$$

From Girsanov's Theorem follows that a Radon-Nikodym derivative of this form can be obtained by setting $\kappa(t) = -\sigma(T-t)$. Hence, under the measure Q_T the process

$$W^T(t) = W^*(t) - \int_0^t (-\sigma)(T-s)\,ds \tag{67}$$

is also a Brownian Motion. From this we obtain that for $t < T$ the spot interest rate r follows the process

$$dr = \big(\theta^*(t) - \sigma^2(T-t)\big)\,dt + \sigma\,dW^T \tag{68}$$

under the probability measure Q_T. Given this specification of the process, we can now calculate (again) an expression for the forward rates $f(t,T)$ using (59). We obtain immediately that

$$\begin{array}{l} f(t,T) = r(t) + \int_t^T \big(\theta^*(s) - \sigma^2(T-s)\big)\,ds = r(t) + \\ \int_t^T \theta^*(s)\,ds - \frac{1}{2}\sigma^2(T-t)^2. \end{array} \tag{69}$$

This expression is equal to (54) derived before. It is clear from this example, that the Change of Numeraire Theorem can provide us with a powerful tool to determine prices of interest rate derivatives in a simple fashion.

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