Applications of Monte Carlo/Quasi-Monte Carlo Methods in Finance: Option Pricing

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Abstract. The pricing of options is a very important problem encountered in financial markets today. The famous Black-Scholes model provides explicit closed form solutions for the values of certain (European style) call and put options. But for many other options, either there are no closed form solutions, or if such closed form solutions exist, the formulas exhibiting them are complicated and difficult to evaluate accurately by conventional methods. In this case, Monte Carlo methods may prove to be valuable.

In this paper, we illustrate two separate applications of Monte Carlo and/or quasi-Monte Carlo methods to the pricing of options: first, the method is used to estimate multiple integrals related to the evaluation of European style options; second, an adaptive Monte Carlo method is applied to a finite difference approximation of a partial differential equation formulation of a class of finance problems. Some of the advantages in using the Monte Carlo method for such problems are discussed.

1 Introduction

Options have been widely traded since the creation of an organized exchange in 1973. Monte Carlo methods have been applied to option pricing (see [3]), but these applications seem mainly to be restricted to European style options. The idea is that the value of the option is formulated in terms of the expectation of some random variable (for example, as in formulas like (11) given below); then the average of independent samples of this random variable is used to estimate the value of the option (see [10]). Recently, Monte Carlo methods have also been applied to American style options (see [20]). Besides the applications in option pricing, Monte Carlo methods have also been widely applied to other financial problems (see, e.g. [10]).

In this paper, we first show how Monte Carlo and/or quasi-Monte Carlo methods may be used to estimate multiple integrals related to the evaluation of European style options. Then we apply an adaptive Monte Carlo method (i.e., one in which successively improving approximations to the solution are employed in the simulation) to a finite difference approximation of a partial differential equation formulation arising in a class of finance problems.

The latter method can be applied to options other than European style. Furthermore, this application has the advantage that it can avoid stability and convergence problems that can arise when using conventional finite difference methods (see [12] and Section 4 of this paper).

2 Preliminaries

First, we introduce some terms used throughout this paper; more detailed information can be found in [4], [5], [9], [10] or [12].

1. A European call option is a contract such that the owner may (without obligation) buy some prescribed asset (called the underlying) S at a prescribed time (expiry date) T at a prescribed price (exercise or strike price) E.
2. A European put option is the same as a call option, except that "buy" is replaced by "sell".
3. An American call or put option is the same as the European call or put, except that the American style options can be exercised at any time during their lifetime, not just at expiry. This feature makes it much harder to evaluate these options.
4. An Asian option is an option whose payoff is dependent on the history of the asset price via an average (arithmetic or geometric) of the price over time.
5. A look-back option is an option whose payoff depends not only on the asset price at expiry, but also on the maximum or minimum of the asset price over some time prior to expiry.
6. Risk neutrality is the characteristic ascribed to an investor who is indifferent with respect to risk. A rigorous definition may be given (see, e.g., [8] or [20]) based on an order relationship on a space of random variables on an appropriately defined probability space. However, the most important use of this notion for us is in its application to the risk-neutral evaluation formula (see equation (11)), for which a separate reference is given.
7. The risk-free interest rate r is an idealized interest rate, usually taken to be that of an appropriate Treasury Bond.
8. The Wiener process (also called Brownian motion) dX is a special type of Markov stochastic process with the following properties: $dX \sim N(0, \sqrt{dt})$, where $N(\mu, \sigma)$ is the normal distribution with mean μ and variance σ^2;

One of the basic problems in option pricing is: given the current price of an asset S, the strike price E, the time to expiry T, the risk-free interest rate r, and the equation that is assumed to control the behavior of S as a function of time t:

$$dS = \mu S dt + \sigma S dX \tag{1}$$

where dX is a Wiener process , μ (a measure of the average rate of growth of the asset price) is the drift rate and σ is the volatility of the asset (characterizing fluctuations in the price S), how may one determine a "fair" current value $V(S,t)$ of the option?

The well-known Black-Scholes model for a European call option can be described ([2] or [12]) by the following (diffusion-type) partial differential equation for this value:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2}$$

with final condition

$$V(S,T) = \max(S - E, 0), \tag{3}$$

and boundary conditions

$$V(0,t) = 0,\ V(S,t) \sim S \text{ as } S \to \infty. \tag{4}$$

The European put option satisfies the same equation as (2), but with final condition

$$V(S,T) = \max(E - S, 0), \tag{5}$$

and boundary conditions

$$V(0,t) = Ee^{-r(T-t)},\ V(S,t) \sim 0 \text{ as } S \to \infty. \tag{6}$$

In both cases, there are explicit closed form solutions. For the call option, the solution is

$$V(S,t) = C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{7}$$

with

$$d_1 = \frac{\ln(\frac{S}{E}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{8}$$

and

$$d_2 = \frac{\ln(\frac{S}{E}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{9}$$

and $N(z)$ is the cumulative distribution function of the standard normal distribution. For the put option,

$$V(S,t) = P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{10}$$

with the same d_1, d_2, and $N(z)$. For most other style options, however, there are no known closed form solutions. Thus, approximate methods and numerical methods, such as lattice methods (see [5]) and finite difference methods (see [12]) are used to estimate their values.

Monte Carlo methods can be very useful in such cases if the solution (i.e., the value, V) can be expressed as the expectation of some random variable(s). This is made possible by the risk-neutral valuation formula [20]:

1. For European options:

$$V(S,t) = E\left(e^{-r(T-t)}h(S(T)) \mid S(t) = S, \mu = r\right) \tag{11}$$

where $E(\cdot)$ is the expectation, $h(S)$ is the payoff function, $h(S) = max(S - E, 0)$ for a call option, $h(S) = max(E - S, 0)$ for a put option, etc. From this formula, the well known Black-Scholes formula [2] can be recovered.

2. For American options: if $h(\cdot)$ satisfies appropriate regularity conditions then

$$V(S,t) = \sup_{\{\tau:\tau\in[0,T],\text{ an optional time}\}} E\left(e^{-r(\tau-t)}h(S(\tau)) \mid S(t) = S, \mu = r\right) \tag{12}$$

where τ is said to be an optional time (or stopping time) with respect to $\{S(t) : t \geq 0\}$ if, for all real $t \geq 0$, $\{\varpi : \tau(\varpi) > t\}$ is completely determined by $\{S(u)$: $0 \leq u \leq t\}$.

3 Multiple Integrals Related to Option Values

Monte Carlo and/or quasi-Monte Carlo methods can be directly applied to finance problems involving multidimensional integrals. For example, Paskov uses a quasi-Monte Carlo sequence – the Sobol sequence - to find the present value of securities which involve up to 360 dimensional integrals; see [11].

We present here another example mentioned in several references ([1], [23]). Consider a European option whose payoff depends on k (≥ 1) assets with prices S_i, $i = 1, ..., k$. Each asset follows the random walk

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i \tag{13}$$

where σ_i is the annualized standard deviation for the i-th asset and dX_i is Brownian motion. Suppose at expiry time T that the payoff is given by $h(S_1', \ldots, S_k')$ (where S' denotes the value of the ith asset at expiry). Then the current value, V, of the option (assuming risk neutrality) will be

$$V = e^{-r(T-t)}(2\pi(T-t))^{-k/2}(\det \Sigma)^{-1/2}(\sigma_1 \cdots \sigma_k)^{-1} \times$$

$$\int_0^\infty \cdots \int_0^\infty \frac{h(S_1', \ldots, S_k')}{S_1' \cdots S_k'} \exp(-\frac{1}{2}\alpha^T \Sigma^{-1} \alpha) dS_1' \cdots dS_k' \tag{14}$$

where $\alpha_i = \left(\sigma_i(T-t)^{1/2}\right)^{-1}\left(\log \frac{S_i'}{S_i} - (r - \frac{\sigma_i^2}{2})(T-t)\right)$, r is the risk-free interest rate and Σ is the covariance matrix (the $(i,j)-$entry is the covariance of dX_i and dX_j) for the k assets. For problems like these, the multiple integrals can be estimated by using either pseudorandom numbers or quasirandom sets/sequences, such as Niederreiter's $(t, m, s)-$ nets and $(t, s)-$sequences ([14]). Furthermore, if the integrands are smooth, lattice rules ([14]) could also be used after periodizing the integrand function.

To achieve the formulation we seek, the infinite domain of integration in (14) can be mapped into the k-dimensional unit hypercube in a variety of ways. For example, $\frac{2}{\pi}\arctan(x)$ maps $(0, \infty)$ to $(0, 1)$. One could also make use of the distribution function of various random variables. Such a mapping transforms the problem to one in which an integral $\int_0^1 \cdots \int_0^1 g(x_1, \cdots, x_k)dx_1 \cdots dx_k$ over the hypercube is sought.

Example 1:

With $k = 5$, $g = exp(-x_1x_2x_3x_4x_5)$ and with appropriate choices of the constants involved in equation (14), we obtain the five dimensional integral

$$I = \int_0^1 \cdots \int_0^1 exp(-x_1x_2x_3x_4x_5)dx_1 \cdots dx_5$$

to estimate. For this problem, we applied both the Monte Carlo method, using pseudorandom numbers to draw samples from the 5-dimensional hypercube, and the low discrepancy Halton sequence based on the first five primes (see [14]). The results are as follows:

Table 1: Errors for multiple integral appearing in pricing of European options

Number of points	Halton sequence	pseudorandom sequence
1,000	5.313269×10^{-1}	2.511778×10^{0}
10,000	2.176994×10^{-1}	5.150753×10^{-1}
100,000	3.348304×10^{-2}	1.425746×10^{-1}
1,000,000	5.446822×10^{-5}	6.340354×10^{-2}

As can been seen from the numerical results, the low discrepancy sequence produces more rapid convergence, and lower errors, than the pseudorandom sequence. This is as anticipated since the pseudorandomly obtained averages converge at the rate $O(N^{-1/2})$, while the quasirandomly obtained averages converge at a rate closer to $O(N^{-1})$, where N is the number of samples used. For larger values of k, the advantage of using either pseudorandom or quasirandom methods in place of more conventional quadrature formulas should become even more pronounced.

4 Finite Difference Methods Implemented with Adaptive Monte Carlo Methods

In this section, we illustrate how Monte Carlo methods can be applied to option pricing problems via the use of finite difference approximations.

The value $V(S,t)$ of a European style option based on a single asset satisfies equation (2) and corresponding final and boundary conditions. Similar equations and final conditions hold for American style options, but with more complicated (free) boundary conditions. At present, there are no explicit general solutions for American style options. To find numerical solutions, the finite difference method is widely used. American style options can be formulated in the Linear Complementary Form, and then any one of many deterministic methods could be applied after using finite difference (explicit or implicit) discretization; see [11].

For the sake of illustrating the use of adaptive Monte Carlo methods, we consider the simple case of a European put option. Using the same notation as for equations (2), (3), (4), we make the changes of variables $S = Ee^x$, $\tau = \frac{1}{2}\sigma^2(T-t)$,

$$V(S,t) = E\exp(-\frac{1}{2}(k_2-1)x - (\frac{1}{4}(k_2-1)^2 + k_1)\tau)u(x,\tau) \qquad (15)$$

where $k_1 = \frac{r}{\frac{1}{2}\sigma^2}$, $k_2 = \frac{r}{\frac{1}{2}\sigma^2}$. The Black-Scholes equation (2) then simplifies to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad (16)$$

with initial and boundary conditions

$$u(x,0) = \max\left(e^{\frac{1}{2}(k_2-1)x} - e^{\frac{1}{2}(k_2+1)x}, 0\right) \qquad (17)$$

$$\lim_{x\to\infty} u(x,\tau) = 0, \quad \lim_{x\to-\infty} u(x,\tau) \sim e^{\frac{1}{2}(k_2-1)x + \frac{1}{4}(k_2-1)^2\tau}. \qquad (18)$$

Solving these equations exactly gives

$$V(S,t) = E^{\frac{1}{2}(1+k_2)} S^{\frac{1}{2}(1-k_2)} \exp\left(-\frac{1}{8}((k_2-1)^2 + 4k_1)\sigma^2(T-t)\right) \times$$
$$u(\log\frac{S}{E}, \frac{1}{2}\sigma^2(T-t)). \qquad (19)$$

Now assume that both δt and δx are small and let $u_n^m = u(n\delta x, m\delta\tau)$, $-N_1 \le n \le N_2$, $0 \le m \le M$, $M\delta\tau = \frac{1}{2}\sigma^2(T-t)$, $\alpha = \frac{\delta\tau}{(\delta x)^2}$. Denote by v_n^m the numerical approximation to the exact value u_n^m that results from any of the standard discretizations of (16) using the time difference δt and spatial difference δx. For example, the general Crank-Nicolson approximation (see [12] for details) results in the system of difference equations

$$(1+2(1-\theta)\alpha)\, v_n^{m+1} - \frac{1}{2}\alpha(1-\theta)(v_{n-1}^{m+1} + v_{n+1}^{m+1}) = (1-2\alpha\theta)v_n^m + \alpha\theta(v_{n-1}^m + v_{n+1}^m), \qquad (20)$$

which can be formulated as a matrix equation

$$C\mathbf{v}^{m+1} = D\mathbf{v}^m + \mathbf{b}^m = \mathbf{z}^m \tag{21}$$

where

$$C = \begin{pmatrix} 1+2(1-\theta)\alpha & -(1-\theta)\alpha & 0 & \cdots & 0 \\ -(1-\theta)\alpha & 1+2(1-\theta)\alpha & -(1-\theta)\alpha & \cdots & 0 \\ 0 & -(1-\theta)\alpha & 1+2(1-\theta)\alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2(1-\theta)\alpha \end{pmatrix}, \tag{22}$$

and

$$D = \begin{pmatrix} (1-2\alpha\theta) & \alpha\theta & 0 & \cdots & 0 \\ \alpha\theta & (1-2\alpha\theta) & \alpha\theta & \cdots & 0 \\ 0 & \alpha\theta & (1-2\alpha\theta) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (1-2\alpha\theta) \end{pmatrix} \tag{23}$$

are $(N_1+N_2-2)\times(N_1+N_2-2)$ matrices,

$$\mathbf{v}^{m+1} = \begin{pmatrix} v_{N_2-1}^{m+1} \\ \vdots \\ v_0^{m+1} \\ \vdots \\ v_{-N_1+1}^{m+1} \end{pmatrix}, \mathbf{b}^m = \begin{pmatrix} \alpha\theta v_{N_2}^m + (1-\theta)\alpha v_{N_2}^{m+1} \\ 0 \\ \vdots \\ 0 \\ \alpha\theta v_{-N_1}^m + (1-\theta)\alpha v_{-N_1}^{m+1} \end{pmatrix} \tag{24}$$

are $N_1 + N_2 - 2$ dimensional vectors, and $\theta \in [0, 1]$ is a parameter. With $\theta = 1$, 0 and 0.5, (21) corresponds to the explicit method, fully implicit method and the Crank-Nicolson method, respectively. The coefficient matrix C is invertible since it is diagonally dominant.

To find the option values, we use the initial condition (17) to find $v_n^0 \approx u_n^o$, -$N_1 \leq n \leq N_2$, then use (18) and (21) to find $v_n^1, \cdots, v_n^M$ (-$N_1 \leq n \leq N_2$). This gives us approximate values v_n^m which are substituted into equation (19) (possibly using linear interpolation, if necessary, to compute the first argument in $u(\cdot,\cdot)$).

One could, of course, use a variety of deterministic methods for solving the matrix equations (21). For example, in the one dimensional case under consideration, a simple LU decomposition would be very effective. However, when there is more than one spatial variable and when the order of the matrix is large and/or the matrix is not as sparse, the adaptive Monte Carlo method we now briefly describe might be preferred to the use of, say, iterative deterministic methods such as the method of successive overrelaxation [12].

To solve (21) for $\mathbf{v}^{m+1}$ using adaptive Monte Carlo methods, let

$$H = I - C = \begin{pmatrix} -2(1-\theta)\alpha & (1-\theta)\alpha & 0 & \cdots & 0 \\ (1-\theta)\alpha & -2(1-\theta)\alpha & (1-\theta)\alpha & \cdots & 0 \\ 0 & (1-\theta)\alpha & -2(1-\theta)\alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2(1-\theta)\alpha \end{pmatrix},$$

so that (21) becomes

$$\mathbf{v}^{m+1} = H\mathbf{v}^{m+1} + \mathbf{z}^{m}. \tag{25}$$

Adaptive Monte Carlo methods for matrix problems originated with Halton [16] and have been reexamined and extended recently (see [17], [18] and [19] Chapter 4). For the solution of equations such as (25), they are based on the construction of a set of random walks on the integers from 1 to N, where N is the number of components of the solution vector. A matrix of transition probabilities, defined in terms of the coefficient matrix H, is used to move each random walk from one discrete state to another. For example, we could define the transition matrix by

$$p_{ij} = \begin{cases} \frac{1}{4}, & i = j \\ \frac{1}{8}, & |i-j| = 1 \\ 0, & \text{otherwise} \end{cases}$$

This makes the choice independent of both α and θ.

A variety of random variables can be defined on the resulting set of discrete random walks, each of which is an unbiased estimator of a given component of the solution vector. When the random walks are processed in finite batches, called stages, and when a learning algorithm is imposed, whereby information from the kth stage is used to improve the estimator used in stage $k+1$, very rapid convergence can be achieved. In fact, in [17], [18] and [19] Chapter 4, geometric convergence is obtained based on two quite different algorithms: one that uses correlated sampling to achieve the learning, and a second based on importance sampling. We applied the first of these algorithms to the solution of the problem described in equation (25).

Example 2:

We chose $E = 10$, $T = 0.5$, $\sigma = 0.2$, $r = .05$, $\alpha = 0.5$ and $.75$, M (number of time steps) $= 50$, $N_1 + N_2 - 2$ (number of spatial intervals) $= 201$. We solved equation (25) for each value of m using several different methods: the explicit method, the implicit method, the Crank-Nicolson method, and our adaptive Monte Carlo method (for which we set $\theta = 0.5$ in equation (21)). We obtained the following numerical results, where the error is defined as the difference between the values obtained by the given method and those obtained by the Black-Scholes formula. We also compared the option values based on our approximate solutions with the exact values computed by the Black-Scholes formula (10). The results are shown in Tables 2-5.

We see that the Monte Carlo method provides an alternative way to solve the matrix equation (21). Furthermore, it is well known (see [12]) that the explicit method is not stable for $\alpha > 0.5$, and evidence of this unstability can be seen in Table 3 (notice the negative and unreasonably large values appearing in the column labeled "explicit" there).

Table 2: Option value comparison with $\alpha = 0.50$

Asset price	explicit	implicit	Crank-Nicolson	Monte Carlo	Black-Scholes
2.0	7.758119	7.759626	7.758868	7.753064	7.753099
3.0	6.753000	6.752949	6.752974	6.752717	6.753099
4.0	5.752915	5.752847	5.752881	5.752432	5.753099
5.0	4.752684	4.752597	4.752640	4.752088	4.753099
6.0	3.752553	3.752528	3.752537	3.752882	3.753099
7.0	2.756035	2.756930	2.756527	2.755651	2.753111
8.0	1.798253	1.799422	1.798493	1.798435	1.768465
9.0	0.988809	0.987380	0.987519	0.987092	1.210217
10.0	0.439178	0.439353	0.440639	0.439288	0.954219
11.0	0.162013	0.159586	0.160289	0.160245	0.219695
12.0	0.048901	0.048377	0.048215	0.048591	0.012961
13.0	0.012386	0.012809	0.012438	0.012623	0.000259
14.0	0.002668	0.003087	0.002849	0.002899	0.000002
15.0	0.000470	0.000696	0.000593	0.000630	9.8590×10^{-9}
16.0	0.000083	0.000152	0.000116	0.000128	2.6451×10^{-11}

Table 3: Option value comparison with $\alpha = 0.75$

Asset price	explicit	implicit	Crank-Nicolson	Monte Carlo	Black-Scholes
2.0	-2.6725×10^{11}	9.410846	9.411412	7.753064	7.753099
3.0	-8.0199×10^{8}	6.757009	6.756359	6.752717	6.753099
4.0	5.7436×10^{0}	5.752971	5.753004	5.752432	5.753099
5.0	3.4009×10^{0}	4.752723	4.752766	4.752088	4.753099
6.0	2.4973×10^{6}	3.752823	3.752833	3.752882	3.753099
7.0	-2.6294×10^{9}	2.757045	2.756641	2.755651	2.753111
8.0	-4.4210×10^{10}	1.799410	1.798483	1.798435	1.768465
9.0	-3.2824×10^{11}	0.987613	0.987753	0.987092	1.210217
10.0	-2.9109×10^{12}	0.439805	0.441086	0.439288	0.954219
11.0	-1.0840×10^{12}	0.159611	0.160316	0.160245	0.219695
12.0	3.5706×10^{11}	0.048457	0.048297	0.048591	0.012961
13.0	-6.6102×10^{10}	0.012772	0.012401	0.012623	0.000259
14.0	-5.4117×10^{8}	0.003076	0.002837	0.002899	0.000002
15.0	1.6382×10^{8}	0.000684	0.000580	0.000630	9.8590×10^{-9}
16.0	3.2223×10^{6}	0.000147	0.000111	0.000128	2.6451×10^{-11}

Table 4: Option value error comparison with $\alpha = 0.50$

Asset price	explicit error	implicit error	C-N error	MC error
2.0	−0.00502	−0.00653	−0.00577	0.00003
3.0	0.00001	0.00015	0.00013	0.00038
4.0	0.00018	0.00025	0.00022	0.00067
5.0	0.00042	0.00050	0.00046	0.00101
6.0	0.00055	0.00057	0.00056	0.00022
7.0	−0.00292	−0.00382	−0.00342	−0.00254
8.0	−0.02979	−0.03096	0.03003	−0.02997
9.0	0.22141	0.22284	0.222698	0.22312
10.0	0.51504	0.51487	0.51358	0.51493
11.0	0.05768	0.06011	0.05941	0.05945
12.0	−0.03594	−0.03542	−0.03525	−0.03563
13.0	−0.01213	−0.01255	−0.01218	0.01236
14.0	−0.00267	−0.00308	−0.00285	−0.00290
15.0	−0.00047	−0.00070	−0.00059	−0.00063
16.0	−0.00008	−0.00015	−0.00012	−0.00013

Table 5: Option value error comparison with $\alpha = 0.75$

Asset price	explicit error	implicit error	C-N error	MC error
2.0	2.672549×10^{11}	−1.657747	−1.658313	0.00003
3.0	8.019933×10^{8}	−0.003910	−0.003260	0.00038
4.0	9.492120×10^{-3}	0.000128	0.000095	0.00067
5.0	1.352206×10^{0}	0.000376	0.000333	0.00101
6.0	-2.497318×10^{6}	0.000276	0.000266	0.00022
7.0	2.624920×10^{9}	−0.003934	−0.003530	−0.00254
8.0	4.421013×10^{10}	−0.030945	−0.030018	−0.02997
9.0	3.282377×10^{11}	0.2226036	0.222464	0.22312
10.0	2.910941×10^{12}	0.514414	0.513133	0.51493
11.0	1.083964×10^{12}	0.060084	0.059379	0.05945
12.0	-3.570554×10^{11}	−0.035496	−0.035336	−0.03563
13.0	6.610245×10^{10}	−0.012514	−0.012143	0.01236
14.0	5.411707×10^{8}	−0.003074	−0.002835	−0.00290
15.0	-1.638214×10^{8}	−0.000684	−0.000580	−0.00063
16.0	-3.222870×10^{6}	−0.000147	−0.000111	−0.00013

As we observed earlier, conventional deterministic methods can be applied to solve (21) or (25). But these methods may suffer some stability or convergence problems for some values of α; details are given in [11]. However, no such limitations appear for our Monte Carlo methods, since we can make the transition probability matrix P independent of α.

The greater the required accuracy, the larger N_1, N_2 and M should be, and as the order of the matrix increases, the adaptive Monte Carlo method becomes more advantageous when compared even with the best currently available deterministic methods, as pointed out by Halton in [15].

For options (or other derivative securities) based on k (≥ 2) underlying assets S_i satisfying (1), the value V of the option satisfies the equation (see [11])

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{k} S_i \frac{\partial V}{\partial S_i}(\mu_i - \lambda_i \sigma_i) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} = rV \qquad (26)$$

together with suitable final and boundary conditions, where ρ_{ij} is the correlation between dX_i and dX_j, and the remaining parameters are as before. To obtain a numerical solution, the finite difference method may again be applied, obtaining a matrix equation like (21) which is no longer tridiagonal. Once again, adaptive Monte Carlo should provide a very effective approach.

5 Summary and Conclusions

In this paper we have demonstrated that Monte Carlo and/or quasi-Monte Carlo methods can provide useful solutions to many problems arising in finance. When the problem is formulated as a multiple integral in many dimensions, it is well accepted that such methods are often much superior to more conventional deterministic quadrature formulas. As well, when the problem is formulated as a partial differential equation, the use of finite difference approximations leads to a matrix problem which can be solved very efficiently with recently developed adaptive Monte Carlo methods.

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