Credit Spread Specification and the Pricing of Spread Options

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Abstract

This paper presents a simple approach to the pricing of options on spread and some arguments in favor of modelling the spread using its two components instead of the spread itself. We show that, even in a simple Gaussian setting, the spread should not be modelled directly, and that convergence speeds of the two components are crucial parameters. There exist conditions, discussed in this paper, under which the analysis can be reduced to a two-factor model based on the dynamics of the spread itself. Hence, we propose a three-factor model based on the dynamics of the riskless rate and of the two components of the spread. This is done by following the Longstaff (1990) methodology and with the assumption that both the riskless rate and the spread or its two components follow correlated Ornstein-Uhlenbeck processes. Greeks analysis shows that spread options have some very specific features compared to the Black-Scholes-Merton (1973) option model. Moreover, the results show that the mispricing is important and not systematic when one chooses a spread option model based on the dynamics of the spread instead of using the dynamics of its two components. We finally show how the spread option model allows us to price other yield derivatives, like options to exchange a yield for another or the options on the maximum or the minimum of two yields.

Keywords: Credit spread, Option valuation, Change of probability measure

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CREDIT SPREAD SPECIFICATION AND THE PRICING OF SPREAD OPTIONS

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EXECUTIVE SUMMARY

In the early nineties a new generation of derivative instruments, based on credit and currency risks, has emerged. Among other products, the market for spread options has expanded rapidly despite little theoretical knowledge of them. Spread options are derivative instruments whose payoff depends on the spread between two different assets or variables. Spread options are now extensively used in interest rate risk management and have been largely developed. Some of the best known spread options, introduced by Goldman Sachs, are the SYCURVE (options on the slope of a yield curve), the MOTTO (the spread between mortgage and treasury securities) and the ISO (spread between domestic and foreign fixed income securities). Hence, spread options are by definition an interesting tool for managing credit risk as well as domestic-foreign interest-rate risk.

However, despite the rapid growth of the spread option market, which still remains mainly over-the-counter (OTC), the valuation of spread options has yet to be well developed. Existing models are not well suited to the case of interest rate spread. Margrabe (1978) and Ravindran (1993) have proposed to valuing spread options in a Black-Scholes-Merton (1973) setting, which implies that the short-term interest rate has to be constant and that the spread must be given by the difference between two log-normally distributed traded assets. Hence, relying on more realistic assumptions on the spread and the interest rate dynamics, Longstaff and Schwartz (1995) have suggested to modelling interest rate spread options under the assumption that the spread is a non-traded asset, log-normally distributed and correlated with a stochastic short-term interest rate.

This paper will take the Longstaff and Schwartz (1995) proposition one step further by modelling directly the components of the spread instead of the spread itself. By assuming that the components of the spread and the short-term interest rate dynamics are given by three correlated mean-reverting Ornstein-Uhlenbeck processes, we are able to provide a new analytical formula.

A numerical analysis illustrates the main properties of interest rate spread options. It first shows that their value can be below the intrinsic option value, contrarily to the classic options on stock. Further, the Vega, i.e. the sensitivity of the spread option to the spread volatility is not necessarily positive and

monotonic, while an increase in the correlation coefficient between the two spread components still yields a decrease of the option price.

We show that a three-factor model allows for more flexibility than a two-factor model, such as the Longstaff and Schwartz model, but we also show how it can be "dangerous" to reduce the spread specification to a single variable. By not specifying the dynamics of the spread components, one looses a non-negligible quantity of information contained in the convergence speed of the two variables. In other terms, a spread of two percent between four and six percent is radically different from a spread of two percent between ten and twelve percent. This straightforward observation yields a strong difference in spread options prices, if one utilizes a two-factor model instead of a three-factor model. Thus, we also propose a two-factor model, based on the assumption that the spread follows an Ornstein-Uhlenbeck process. We next compare this new model numerically to the three-factor model and show that the mispricing can be huge and not systematic.

In conclusion, we emphasize that even in a simple Gaussian setting, the spread specification may not be reduced to a single-variable problem. There is sufficient information contained in the convergence speed to suggest modelling the two spread components instead of the spread itself in order to price any kind of interest-rate spread options.

1. INTRODUCTION

Interest in spread options has rapidly increased since the beginning of the nineties. They were introduced by Goldman Sachs, who developed several spread options such as the SYCURVE options, which are options on the slope of a yield curve, the MOTTO options, which are options on the spread between mortgage and Treasury securities, or the ISO options, which are options on the spread between foreign fixed-income securities and other foreign fixed-income securities. These examples highlight the great interest in spread options because they are useful in managing two different sources of risk: exchange rate risk and credit risk. Financial markets are increasingly interested in credit risk, and banks as well as financialinstitutions are looking for credit risk hedging instruments. Hence, when the spread is calculated as the difference between a corporate bond yield and a government bond yield, spread options often prove a useful instrument for credit risk management. Moreover, when the spread is based on two foreign fixed-income securities, as in the case of ISO options, spread options can serve to hedge a position in a diffswap.

However, if several spread option valuation models are proposed, their effectiveness in terms of credit spread or foreign/domestic interest rate spread is brought into question. Since the early work of Margrabe [1978], two streams of research have appeared. The first stream of research, initiated by Margrabe [1978] and followed by Ravindran [1993] among others, is based on a Black-Scholes-Merton [1973] setting: they assume that the spread is the difference between two traded assets following log-normal processes (geometric Brownian motion) and that the riskless interest rate is constant. These hypotheses cannot be justified in the case of a spread between two yields. Empirical studies tend to show that a mean-reverting process is, in general, more sustainable than a geometric Brownian motion to model the dynamics of the interest rates. Furthermore, yields are not traded assets and it is di¢cult to suppose that the riskless rate is constant while the risky rate varies through time. Hence, a second strand of research, pioneered by Longstaffand Schwartz [1995], assumes that the spread is not a traded asset and that the riskless rate is stochastic, possibly correlated with the spread. Nevertheless, both approaches tend to model the spread directly without taking into consideration its two components; no one has yet to ask what the consequences of this assumption would be.

Hence, the goal of this paper is first to propose an original analytical formula for spread options, based on more realistic hypotheses, assuming that the spread is composed by two non-traded and mean-reverting components and that the riskless rate is also mean-reverting, stochastic and possibly correlated with the spread components. It allows us to investigate the impact on pricing of the characterization choice of the spread. By means of a further analytical formula where the spread is modelled directly, we also compare numerically the two- and three-factor models. Finally, we show how the three-factor spread option formula enables us to value other credit risk derivatives, such as options on a maximum or a minimum of two yields.

The remainder of this paper is organized as follows. In the first section, we show the condition under which a spread can be expressed directly when the dynamics of its two components are given by two correlated Ornstein-Uhlenbeck processes. We then provide in the second section an analytical formula for pricing spread options, based on the processes of the yields, instead of the spread itself. In the third section, we discuss the implications of choosing either the two- or three-factors model in terms of spread options pricing. Finally, we shall extend our results to the valuation of other claims contingent to two yields, such as minimum or maximum options.

2. SPREAD SPECIFICATION AND IMPLICATIONS

This section is devoted to the analysis of the specification choice of the spread. Before selecting a specific process to model the dynamics of the spread or of its two components, one needs to ask whether we can directly model the spread as one variable directly or if it is necessary to take into account its two components. In a Black-Scholes-Merton [1973] setting, Margrabe [1978] has first defined the price of an option to exchange an asset for another, which is equivalent to a spread option with a zero strike. Then Ravindran [1993] extended Margrabe's results by allowing the spread to take any positive values. Since both used a geometric Brownian motion to describe the dynamics of the underlying assets, their methods are useful in the case of a spread between two stocks. However, the hypothesis of mean-reverting process, such as the Ornstein-Uhlenbeck process, is more appropriate in the case of a spread between two yields. Following a geometric Brownian motion, a yield is supposed to grow exponentially, whereas a mean-reverting process implies that the yield has a tendency to vary towards its long-term mean. If the current value of the yield lies above its long-term mean, the yield should decrease, at least in expectation and if the current value of the yield is below its long-term mean, it should increase. Thus, the Longstaff and Schwartz [1995] approach seems to be more accurate to price yield spread options. Longstaff and Schwartz [1995] applied an Ornstein-Uhlenbeck process to the logarithm of a credit spread between a number of Moody's bond indexes and the long-term US treasury bond yield. It is clear that the use of the spread as a variable in itself simplifies the spread options' model and decreases the number of parameters to estimate. However, we argue that the spread can not be reduced to one variable and that Longstaff and Schwartz [1995], by modelling the spread itself, lose important information. So, the core question is in when can the dynamics of the two yields be replaced by the spread itself? We provide a necessary and su¢cient condition, when the yields follow two Ornstein-Uhlenbeck processes.

Denote the two yields respectively by X_t and Y_t. Let us assume that their dynamics are given by two Ornstein-Uhlenbeck processes:

$$dX_t = \alpha_X (\beta_X - X_t)\, dt + \sigma_X dW_t^1 \tag{2.1}$$

$$dY_t = \alpha_Y (\beta_Y - Y_t)\, dt + \sigma_Y dW_t^2 \tag{2.2}$$

W_t^1 and W_t^2 are two correlated Brownian motions with correlations $\rho_{X;Y}$ given by

$$\rho_{X;Y}\, dt = dW_t^1 dW_t^2 \tag{2.3}$$

W_t^1 and W_t^2 are defined on a probability space $(\Omega; \mathcal{F}; P)$. The information structure is given by the filtration $\{\mathcal{F}_t; 0 \leq t \leq T\}$, generated by the Brownian motions W_t^1 and W_t^2. Let S be the yield spread defined as $(X - Y)$. Thus, the dynamics of S are given by:

$$dS_t = (\alpha_X \beta_X - \alpha_Y \beta_Y - (\alpha_X X - \alpha_Y Y))\, dt + \sigma_X dW_t^1 - \sigma_X dW_t^2 \tag{2.4}$$

Definition 2.1. Let Z_t be defined on the probability space $(\Omega; \mathcal{F}; P)$ such that:

$$Z_t = \left(\sigma_X W_t^1 - \sigma_Y W_t^2\right) / \sigma_S \tag{2.5}$$

with $\sigma_S = \sqrt{\sigma_X^2 + \sigma_Y^2 - 2\rho_{X;Y}\sigma_X\sigma_Y}$

One can verify that Z_t respects the usual properties of Brownian motions, i.e. Z_t is a martingale, has a continuous path and $dZ_t dZ_t = dt$.

Equation (2.4) can be simplified to:

$$dS_t = (\alpha_X \beta_X - \alpha_Y \beta_Y - \alpha_X S - (\alpha_X - \alpha_Y) Y)\, dt + \sigma_S dZ_t \tag{2.6}$$

or

$$dS_t = \alpha_X \left(\beta_S - S - \left(\frac{\alpha_X - \alpha_Y}{\alpha_X}\right) Y\right) dt + \sigma_S dZ_t \tag{2.7}$$

with $\beta_S = \frac{\alpha_X \beta_X - \alpha_Y \beta_Y}{\alpha_X}$

Obviously, equation (2.7) can also be rewritten so that S depends on X instead of Y. If the convergence speeds of the yields differ, the infinitesimal variation of S is directly linked to the current value of Y. It leads to the following result:

Proposition 2.2. Let X and Y be given by (2.1) and (2.2). The spread S, defined as $(X - Y)$, follows an Ornstein-Uhlenbeck process if and only if $\alpha_X = \alpha_Y$.

Then, if the convergence speeds of X and Y differ, the dynamics of the spread cannot be explained only by the spread. Knowledge of one of the two yields is required. In short, we lose important information by taking into account the spread alone. The following example shows the importance of the convergence speeds:

Example 2.3. We have assumed that the yields follow Ornstein-Uhlenbeck processes. This induces that X and Y are mean-reverting towards their long-term means, resp. β_X and β_Y, with convergence speeds, α_X and α_Y. Let assume that: $\beta_X = 0.08$, $\beta_Y = 0.04$, $\alpha_X = 0.1$ and $\alpha_Y = 0.7$. The spread would vary towards 4%. If the current level of X and Y are 0.08 and 0.07 respectively, the spread is equal to 1%. Moreover, since X is equal to its long term mean and Y has a high adjustment speed relative to X, the spread should increase rapidly, at least in expectation. Suppose now that X and Y are equal to 0.05 and 0.04 respectively. The spread is still equal to 1% but will converge to 4% at a lower speed since in expectation, Y should not vary whereas X should increase albeit slowly.

The proposition (2.2) is true in the case of Ornstein-Uhlenbeck processes but cannot be directly extended to a larger class of processes. For example, the equality is a necessary condition in the case of the Cox, Ingersoll and Ross [1985] process, but the square root form of the diffusion term implies that it is not sufficient. It must have an important impact in terms of the pricing of claims contingent on yield spreads. Moreover, this impact should be tempered for long-term options since the yields are expected to converge to their long-term mean, whatever their current level. Thus, knowing the level of the spread is not enough to determine its future path. A spread option model based only on the dynamics of the spread therefore will be less precise than a model using the dynamics of its two components, unless the convergence speeds are equal.

3. A NEW THREE-FACTOR MODEL

As we have shown in the previous section, it is important to take into account the dynamics of each component of the spread rather than to rely on a simplified process for the spread itself. In this section, we thus provide a new analytical formula based on the dynamics of X and Y.

3.1. Valuation

As in the previous section, we assume that the dynamics of the two yields included in the spread, noted X and Y, are given by Ornstein-Uhlenbeck processes. Furthermore, we suppose that the riskless interest rate r; also follows an Ornstein-Uhlenbeck process. Thus:

$$dX_t = \alpha_X (\beta_X - X_t)\, dt + \sigma_X dW_t^1 \tag{3.1}$$

$$dY_t = \alpha_Y (\beta_Y - Y_t)\, dt + \sigma_Y dW_t^2 \tag{3.2}$$

$$dr_t = \alpha_r (\beta_r - r_t)\, dt + \sigma_r dW_t^3 \tag{3.3}$$

where $W_t^1; W_t^2;$ and W_t^3 are three correlated Brownian motions.

Let $C(r_t; X_t; Y_t; K; t; T)$ be the value of a call on the spread $(X - Y)$ at time t with maturity T and strike K. Contrary to the models of Margrabe [1978] or Ravindran [1995], the strike can be either negative, zero or positive. By applying the Ito's Lemma on $C(r_t; X_t; Y_t; K; t; T)$ and by creating a portfolio consisting of four spread options with different maturities such that it is risk-free, we obtain that $C(r_t; X_t; Y_t; K; t; T)$ must satisfy the following partial differential equation (thereafter PDE):

$$\begin{aligned} C_t &+ (\alpha_r (\beta_r - r_t) - \sigma_r \lambda_r) C_r + (\alpha_X (\beta_X - X_t) - \sigma_X \lambda_X) C_X \\ &+ (\alpha_Y (\beta_Y - Y_t) - \sigma_Y \lambda_Y) C_Y + \frac{1}{2} C_{rr} \sigma_r^2 + \frac{1}{2} C_{XX} \sigma_X^2 + \frac{1}{2} C_{YY} \sigma_Y^2 \\ &+ C_{Xr} \rho_{rX} \sigma_r \sigma_X + C_{XY} \rho_{XY} \sigma_Y \sigma_X + C_{Yr} \rho_{rY} \sigma_r \sigma_Y - rC = 0 \end{aligned} \tag{3.4}$$

with the boundary condition:

$$C(r_t; X_t; Y_t; K; t; T) = Max(X_T - Y_T - K; 0) \tag{3.5}$$

where $\lambda_r; \lambda_X$ and λ_Y are the risk premia associated with the three interest rates, which are assumed to be constant, and K is the option strike.

Applying Lemma 3 of Cox, Ingersoll and Ross [1985], we know that any contingent claim $C(r_t; X_t; Y_t; t; T)$ satisfying the PDE (3.4) with the boundary condition (3.5) has the following solution:

$$C(r_t; X_t; Y_t; K; t; T) = \widehat{E}\left(\exp\left(- \int_t^T r_u du \right) Max(X_T - Y_T - K) \right) \tag{3.6}$$

The circumflex above the expectation term stands for the risk-adjusted probabilities. Therefore, following equation (3.6), the price of a call on spread is equal to the expected value of the payoff times a discount factor, under the risk-adjusted probabilities. Because of the correlation between the discount factor and the payoff, due to the correlation between r, X and Y, it is difficult to solve equation (3.6) directly. Two similar yet distinctive methods exist to solve this problem; that of Longstaff [1990] and Jamshidian [1990], which were initially applied to the single factor problem. The Longstaff [1990] methodology consists in separating the expected discounted payoff into two components, the price of a discount bond $B(r_t;t;T)$ multiplied by a function $G(X_t;Y_t;t;T)$ still to be defined.

$$C(r_t;X_t;Y_t;t;T) = B(r_t;t;T)\,G(X_t;Y_t;t;T) \tag{3.7}$$

Since the riskless interest rate follows an Ornstein-Uhlenbeck process, $B(r_t;t;T)$ is the price of a discount bond using the Ornstein-Uhlenbeck [1977] formula. Hence, its PDE and value are given by:

$$B_t + (\alpha_r(\beta_r - r_t) - \sigma_r\lambda_r)B_r + \frac{1}{2}B_{rr}\sigma_r^2 - rB = 0 \tag{3.8}$$

$$B(r_t;t;T) = De^{-r_tF} \tag{3.9}$$

with $F = \frac{1-e^{-\alpha_r(T-t)}}{\alpha_r}$ and $D = \exp\left[\left(\beta_r + \frac{\sigma_r\lambda_r}{\alpha_r} - \frac{\sigma_r^2}{2\alpha_r}\right)(F-(T-t)) - \frac{\sigma_r^2F^2}{4\alpha_r}\right]$

$G(X_t;Y_t;t;T)$ is a function still to be defined but does not depend on r_t. From the PDE of $C(r_t;X_t;Y_t;t;T)$ and $B(r_t;t;T)$; we can easily deduce the PDE that must be satisfied by $G(X_t;Y_t;t;T)$:

$$\begin{aligned}
&G_t + (\alpha_X(\beta_X - X_t) - \sigma_X\lambda_X - \rho_{rX}\sigma_r\sigma_X F)G_X \\
&+ (\alpha_Y(\beta_Y - Y_t) - \sigma_Y\lambda_Y - \rho_{rY}\sigma_r\sigma_Y F)G_Y + \frac{1}{2}G_{rr}\sigma_r^2 + \frac{1}{2}G_{XX}\sigma_X^2 + \frac{1}{2}G_{YY}\sigma_Y^2 \\
&+ G_{Xr}\rho_{rX}\sigma_r\sigma_X + G_{XY}\rho_{XY}\sigma_Y\sigma_X + G_{Yr}\rho_{rY}\sigma_r\sigma_Y = 0
\end{aligned} \tag{3.10}$$

with the boundary condition:

$$G(X_T;Y_T;T;T) = \frac{C(r_t;X_t;Y_t;T;T)}{B(r_t;T;T)} = \mathrm{Max}(X_T - Y_T - K;0) \tag{3.11}$$

From theorem 5.3 of Friedman [1975], the unique solution to (3.10) under boundary condition (3.11) is:

$$G(X_t, Y_t, t, T) = \hat{E}\left(Max\left(X_t - Y_t - K, 0\right)\right) \tag{3.12}$$

where ($\hat{E}$) stands for the computation of the expectation under the following modified processes:

$$\begin{aligned} dX_t &= \alpha_X \left(\beta_X - \frac{\sigma_X \lambda_X}{\alpha_X} - \frac{\rho_{rX}\sigma_r\sigma_X F}{\alpha_r} - X_t \right) dt + \sigma_X dW_t^1 \qquad (3.13) \\ &= \alpha_X \left(\beta'_X - X_t \right) dt + \sigma_X dW_t^1 \end{aligned}$$

$$dY_t = \alpha_Y \left(\beta_Y - \frac{\sigma_Y \lambda_Y}{\alpha_Y} - \frac{\rho_{rY}\sigma_r\sigma_Y F}{\alpha_r} - Y_t \right) dt + \sigma_Y dW_t^2 \tag{3.14}$$

$$= \alpha_Y \left(\beta'_Y - Y_t \right) dt + \sigma_Y dW_t^2 \tag{3.15}$$

Although the diffusion term of each process is not affected by this change of probability, the long-term mean is corrected by two time-varying but deterministic components: $\frac{\sigma_X \lambda_X}{\alpha_X}$, since equation (3.6) is taken under the risk-adjusted probabilities, and $\frac{\rho_{rX}\sigma_r\sigma_X F}{\alpha_r}$, which results from the adjustment for the correlation between the payoff and the discount factor. Hence, the price of a contingent claim is equal to the discounted value of the expected payoff, under two modified processes that take into account the correlation between the payoff and the discount factor as well as the risk premia. Moreover, it implies that under these two modified processes, X and Y are still normally distributed, with means and variances equal to:

$$\mu_X = E(X_T / X_t) = \beta'_X + \left(X_t - \beta'_X\right) e^{-\alpha_X (T-t)}$$
$$\mu_Y = E(Y_T / Y_t) = \beta'_Y + \left(Y_t - \beta'_Y\right) e^{-\alpha_Y (T-t)}$$
$$\sigma_X^2 = V(X_T / X_t) = \left[1 - e^{-2\alpha_X (T-t)}\right] (\sigma_X / 2\alpha_X)$$
$$\sigma_Y^2 = V(Y_T / Y_t) = \left[1 - e^{-2\alpha_Y (T-t)}\right] (\sigma_Y / 2\alpha_Y)$$

It becomes straightforward to compute the second piece on the RHS of equation (3.7) and by extension the price of a call on yield spread. Naturally, the price of a Put on spread $P(r_t, X_t, Y_t, K, t, T)$ can be found by using the same method but with a payoff equal to $Max(K - (X_T - Y_T), 0)$.

Proposition 3.1. Let us assume that the dynamics of X, Y and r are given by (3.1), (3.2) and (3.3). Then:

- the price of a call, $C(r_t; X_t; Y_t; K; t; T)$ on the spread $S = (X - Y)$ with strike K is equal to:

$$C(r_t; X_t; Y_t; K; t; T) = B(r_t; t; T)\left((\mu_X - \mu_Y - K)\Phi(d) + \frac{\sigma_{(X-Y)}}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2}\right) \tag{3.16}$$

where $d = \frac{\mu_X - \mu_Y - K}{\sigma_{(X-Y)}}$; $\sigma_{(X-Y)} = \sqrt{(\sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y)}$ and $\Phi(\cdot)$ stands for the cumulative distribution of standard normal distribution

- the price of a put, $P(r_t; X_t; Y_t; t; T)$; on the spread $S = (X - Y)$ is equal to:

$$P(r_t; X_t; Y_t; K; t; T) = B(r_t; t; T)\left((K - (\mu_X - \mu_Y))\Phi(-d) + \frac{\sigma_{(X-Y)}}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2}\right) \tag{3.17}$$

- The Put-Call Parity for a spread option is given by:

$$\begin{aligned} & P(r_t; X_t; Y_t; K; t; T) - C(r_t; X_t; Y_t; K; t; T) \\ = \; & KB(r_t; t; T) + B(r_t; t; T)(\mu_X - \mu_Y) \end{aligned} \tag{3.18}$$

- Since $Max(K - (X - Y); 0)$ is identical to $Max((Y - X) - (-K); 0)$, the value of a put on a spread between X and Y with a strike K is equal to the value of a call on a spread between Y and X with a strike $(-K)$. Thus we can deduce the following Put-Call symmetry:

$$P(r_t; X_t; Y_t; K; t; T) = C(r_t; Y_t; X_t; -K; t; T) \tag{3.19}$$

This Put-Call parity differs from the Black-Scholes-Merton [1973] version. We find a result closer to the Put-Call parity of Longstaff-Schwartz [1995]: the difference between a put and a call with same maturity T and same strike K depends on the discounted forward of the spread instead of its current value, because of the non-tradability of the yields. The Put-Call symmetry is a simple relation, which is true for any spread options, whatever the initial hypotheses on the processes.

Furthermore, the model presented in this section can be easily extended to processes with time-varying but deterministic parameters. Instead of using

Ornstein-Uhlenbeck processes, we can employ the extended Ornstein-Uhlenbeck process, initially proposed by Hull and White [1990], of the form:

$$dX_t = \alpha_X(\beta_X(t) - X_t)\,dt + \sigma_X\,dW_t^1 \tag{3.20}$$

$$dY_t = \alpha_Y(\beta_Y(t) - Y_t)\,dt + \sigma_Y\,dW_t^2 \tag{3.21}$$

$$dr_t = \alpha_r(\beta_r(t) - r_t)\,dt + \sigma_r\,dW_t^3 \tag{3.22}$$

The long-term mean are allowed to vary through time but in a deterministic way. Thus, the methodology previously applied still works in this setting because the conditional distributions of X and Y are still Gaussian even if the β are time-varying but deterministic. Instead of obtaining the bond price by the Ornstein-Uhlenbeck [1977] formula, we obtain the Hull and White [1990] bond formula and the parameters μ_X; μ_Y and $\sigma_{(X-Y)}$ are functions of the time-varying β: This simple extension allows us to fit the initial term structure exactly but it also induces the same problem of practical tractability as in the Hull and White [1990] model. It needs a daily adjustment of the time-varying parameters.

3.2. Greeks

Since the articles of Longstaff [1990] and Longstaff and Schwartz [1995], it is well known that the options on non-traded assets that follow a mean-reverting process have some differing properties from the Black-Scholes-Merton [1973] option model. To illustrate this feature, this section provides an analysis of the Greeks of the option, i.e. the sensitivity of the option to variables and parameters.

3.2.1. Time to maturity

The first result is that the value of a call can lie below its intrinsic value and thus does not respect Theorem 8.3 of Merton [1973]. This theorem states that in order to avoid any arbitrage opportunity, a European call C on a stock A_t that does not pay dividends must respect the inequality:

$$C(A_t; K; t; T) = \mathrm{Max}(A_t - KB(t;T); 0) \tag{3.23}$$

And therefore:

$$C(A_t; K; t; T) = \mathrm{Max}(A_t - K; 0) \tag{3.24}$$

7. APPENDIX

7.1. Proof of proposition 3.1

We have already shown that the price of a call on spread is equal to:

$$C(r_t;X_t;Y_t;t;T) = B(r_t;t;T)\,\tilde{E}\,(Max(X_t - Y_t - K;0)) \tag{7.1}$$

where ($\tilde{E}$) stands for the computation of the expectation under the following modified processes:

$$\begin{aligned} dX_t &= \alpha_X\left(\left(\beta_X - \frac{\sigma_X\lambda_X}{\alpha_X} - \frac{\rho_{rX}\sigma_r\sigma_X F}{\alpha_r}\right) - X_t\right)dt + \sigma_X dW_t^1 \qquad (7.2)\\ &= \alpha_X\left(\beta'_X - X_t\right)dt + \sigma_X dW_t^1 \end{aligned}$$

$$\begin{aligned} dY_t &= \alpha_Y\left(\left(\beta_Y - \frac{\sigma_Y\lambda_Y}{\alpha_Y} - \frac{\rho_{rY}\sigma_r\sigma_Y F}{\alpha_r}\right) - Y_t\right)dt + \sigma_Y dW_t^2 \qquad (7.3)\\ &= \alpha_Y\left(\beta'_Y - Y_t\right)dt + \sigma_Y dW_t^2 \qquad (7.4) \end{aligned}$$

It remains to calculate the second part in the RHS of equation (7.1), i.e. the function $G(X_t;Y_t;t;T)$. By the property of conditional expectations, we have:

$$\begin{aligned} G(X_t;Y_t;t;T) &= \tilde{E}\,(Max(X_t - Y_t - K;0)) \qquad (7.5)\\ &= \tilde{E}_Y\left(\tilde{E}_X\,((Max(X_t - Y_t - K;0))\,|Y)\right) \end{aligned}$$

$$\tilde{E}_X\,((Max(X_t - Y_t - K;0))\,|Y) = \int_{Y+K}^{+\infty}(X - Y - K)\,f(X|Y)\,dX \tag{7.6}$$

where $f(X|Y) = \frac{1}{\sigma_X\sqrt{2\pi}\sqrt{1-\rho^2_{X;Y}}}\exp\left(-\frac{u - v\rho_{X;Y}}{2(1-\rho^2_{X;Y})}\right)$; with $u = \frac{X-\mu_X}{\sigma_X}$ and $v = \frac{Y-\mu_Y}{\sigma_Y}$

After some simplifications, we find:

$$\begin{aligned} \tilde{E}_X\,((Max(X_t - Y_t - K;0))\,|Y) = &\left(\mu_X + \rho_{X;Y}v\sigma_X - Y - K\right)\Phi(-d_1)\\ &+ \frac{\sigma_X\sqrt{1-\rho^2_{X;Y}}}{\sqrt{2\pi}}\exp\left(-0.5d_1^2\right) \end{aligned}$$

with $d_1 = \frac{\frac{Y+K-\mu_X}{\sigma_X} - \rho_{X;Y}}{\sqrt{1-\rho^2_{X;Y}}}$

Then, $G(X_t; Y_t; t; T)$ is equal to:

$$G(X_t; Y_t; t; T) = \int_{-\infty}^{+\infty} \left(\left(\mu_X + \rho_{X;Y} v \sigma_X - Y - K\right) \Phi(-d_1) + \frac{\sigma_X \sqrt{1-\rho^2_{X;Y}}}{\sqrt{2\pi}} \exp\left(-0.5 d_1^2\right) \right) f(Y)\, dY$$

Using the fact that $\int_{-\infty}^{+\infty} f(u)\Phi(a+bu)du = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right)$; this becomes:

$$G(X_t; Y_t; t; T) = \left((\mu_X - \mu_Y - K)\Phi(d) + \frac{\sigma_{(X-Y)}}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2} \right)$$

and consequently the price of a call is equal to:

$$C(r_t; X_t; Y_t; t; T) = B(r_t; t; T) \left((\mu_X - \mu_Y - K)\Phi(d) + \frac{\sigma_{(X-Y)}}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2} \right)$$

16. Mougeot, N., (1997), Evaluation et Couverture d'un Di¤ Swap, Document #9706, IGBF, University of Lausanne.

17. Ravindran, K., (1993), Low-fat Spreads, Risk, October.

18. Vasicek, O., (1977), An Equilibrium Characterization of the Term Structure, Journal of Financial Economics, 5, 177-88.

19. Wei, J., (1994), Valuing Di¤erential Swaps, Journal of Derivatives, Spring, 64-76.

9. FIGURES AND TABLES

	X	Y	S
®	0.3	0.1	0:189707 (5:033$^{E\ ¡\ 3}$)
¯	0.1	0.06	0:039902 (1:97$^{E\ ¡\ 4}$)
¾	0.04	0.03	0:036256 (4:68$^{E\ ¡\ 4}$)

Table 1: Parameters values

(Standard deviation of the estimators are reported in parentheses)

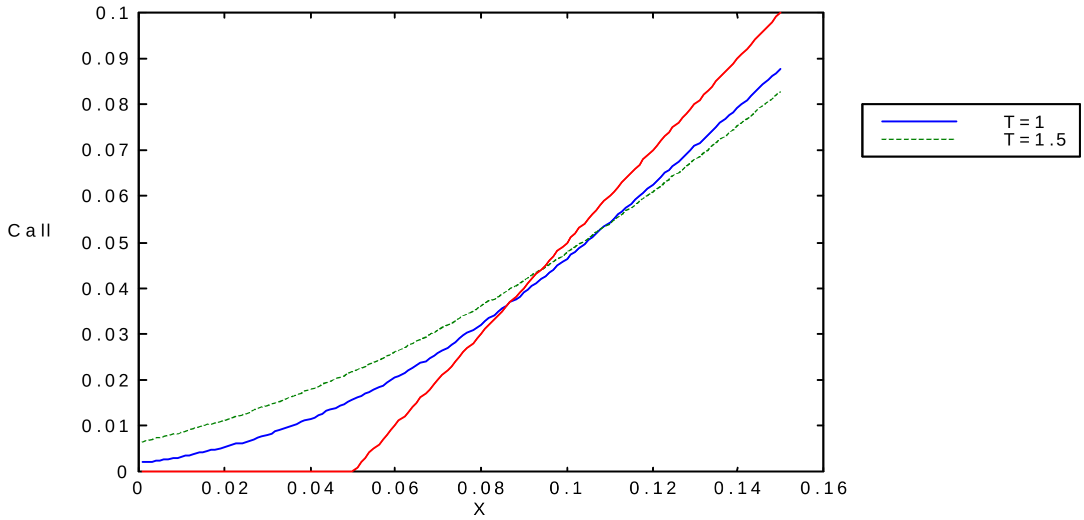

Figure 9.1: Value of a Call with respect to X

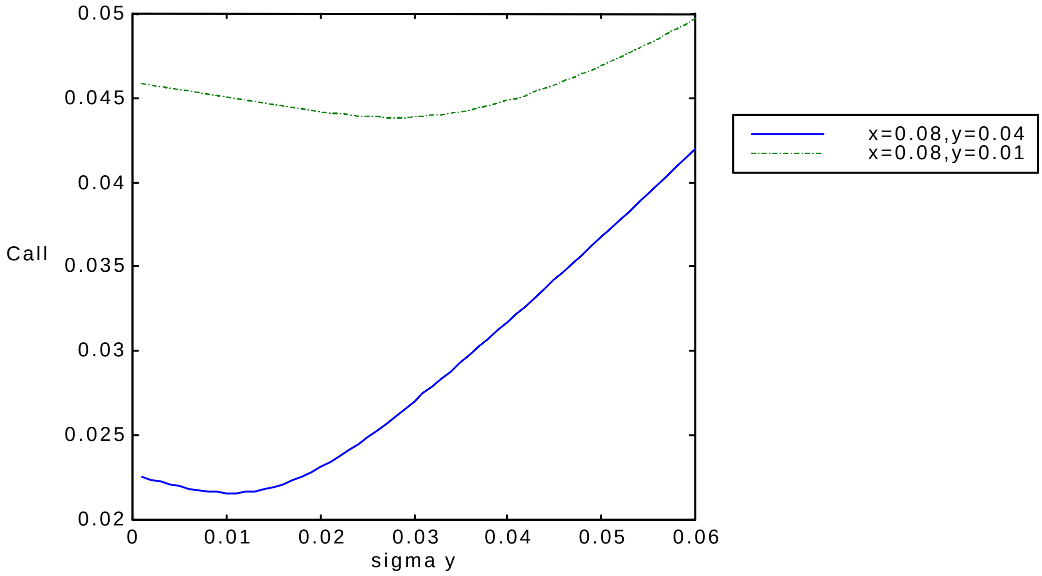

Figure 9.2: Value of a call with respect to σ_Y

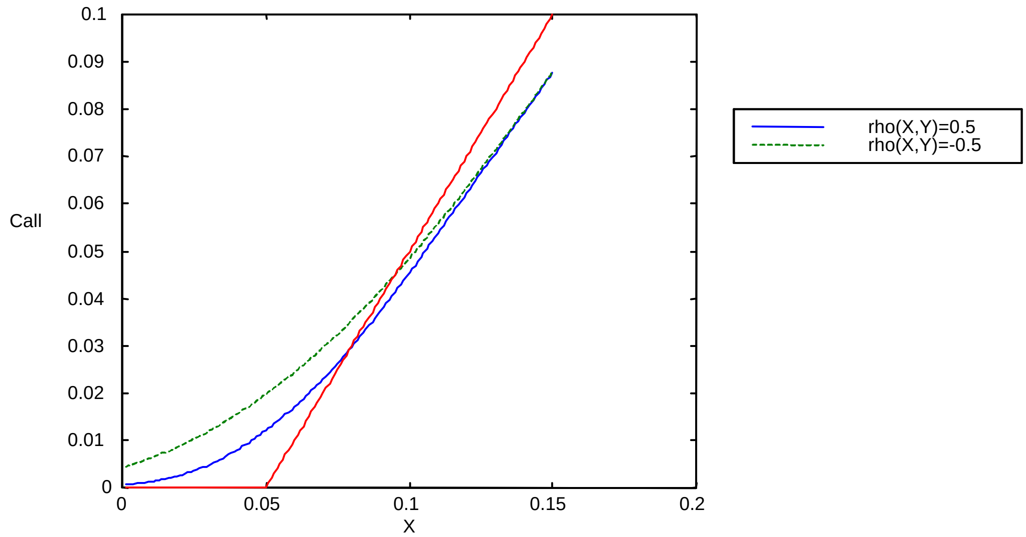

Figure 9.3: Value of a Call with respect to X

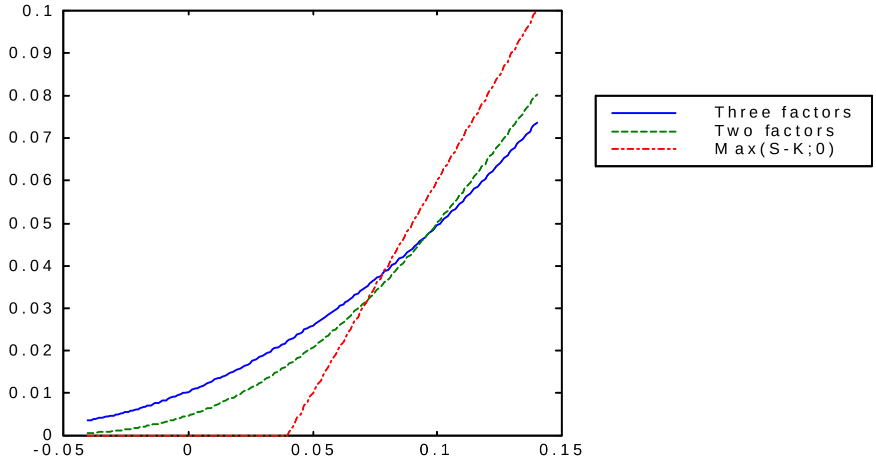

Figure 9.4: Value of a Call on spread with respect to the spread (X varies, Y =0.06, K =0.04)

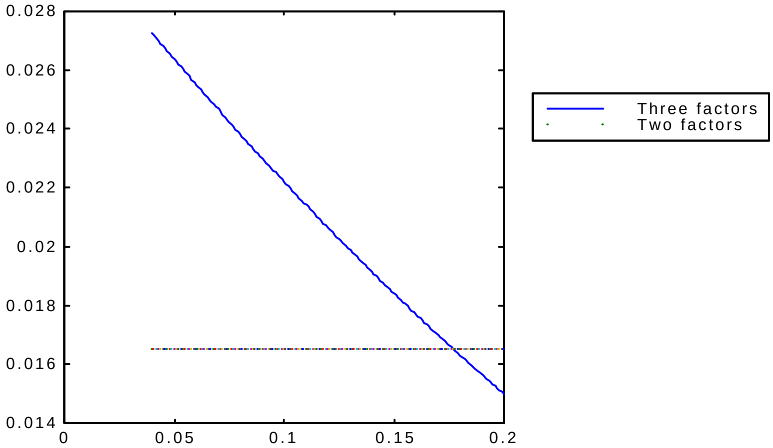

Figure 9.5: Value of a call with respect to X (with X ¡ Y = 0:04, K = 0:04)