Jumping in Line

Claudio Albanese, Sebastian Jaimungal and Dmitri H. Rubisov

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Department of Mathematics, University of Toronto
Math Point Ltd., Toronto
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Abstract

We discuss an efficient numerical method to price derivative claims assuming the underlying follows a jump process of the variance gamma (VG) type. The algorithm is based on the method of lines and involves the solution of ordinary differential equations and an extrapolation scheme similar to Richardson extrapolation. We discuss the method in the context of equity options and show how to price Bermudan and barrier options with VG jumps.

1 Introduction

The Black-Scholes model [1] played a pivotal role in the development of modern pricing theory. Prices in the Black-Sholes model are parameterized in terms of the volatility of the underlying asset and it is well known that discrepancies with market prices are observed. In order to capture the market prices of plain-vanilla options, written on the same underlying, the Black-Sholes volatility parameter must depend on particulars of the contract. These volatilities are known as the implied Black-Sholes volatility and, as a function of strike and maturity, they span the so-called volatility surface. A cross-section of the volatility surface at fixed maturity is termed the volatility smile or skew. One common feature of implied volatilities is that smiles become flatter as maturity lengthens, while as maturity approaches they tend to steepen and implied volatilities rise. Furthermore, out of the money puts are inclined to have higher implied volatilities than out of the money calls. The general increase in the implied volatility can be explained as the market's view that shorter maturity options are more sensitive to shocks in the underlying, while the skewness is the market's view that downward shocks are more likely than upward shocks (an inherent pessimism built into the market).

In the last two decades numerous extensions of the Black-Scholes framework have been proposed. In a strive towards achieving better consistency with both the observed market prices for derivative instruments, and the empirical features of return distributions, various alternatives to geometric Brownian motion have been explored. The proposed models can be conveniently split into two categories: parametric and non-parametric. Parametric models contain a few parameters that serve to adjust the moments of the return distribution and make it asymmetric and kurtotic. Once the model is calibrated, that is the parameters are identified in terms of the price of a few liquid derivative assets, the pricing problem can be addressed. The other category of models, which are non-parametric, resorts to the direct extrapolation of the implied Black-Scholes volatility surface. Based on this intermediate result, a local volatility function can be computed and path dependent options can be valued, see for instance [2, 3, 4]. The proliferation of pricing models is due not only to the strive for numerical efficiency; rather, there is a fundamental issue to be addressed when discriminating between different models. If two models agree on all prices of European style claims at all maturities, they are not necessarily equivalent - they may forecast qualitatively different dynamics of option prices [5], produce different hedge ratios, show discrepancies in the prices of path-dependent options and disagree on what is the optimal exercise policy for American options. Non-parametric diffusion models based on a state dependent volatility function do not produce the correct behaviour. The behaviour of the local volatility is inherited from the particular smile used for calibration, which induces a flattening tendency in the implied volatility as time evolves, contrary to what is observed in the market. In stark contrast to the non-parametric models, the steepening tendency of the volatility smiles at short maturities is fully captured by parametric models based on jump processes or on diffusion processes with stochastic volatility. One of the most promising models, termed the variance gamma (VG) process, was proposed in [6]. That model was demonstrated to capture the most salient features of volatility smiles (an extension of the VG model was also proposed in [7].) The closed form solution derived in [6] for European options allows one to easily calibrate the model; however, for pricing path dependent options, time consuming methods such as Monte-Carlo simulation must be employed. In this paper a new pricing methodology for the VG process is developed which allows for efficient computation of path-dependent options at predefined time intervals.

In this paper we argue that there is a fundamental connection between the approximation scheme known as the methods of lines (MoL) and jump processes. The MoL was first introduced in the financial literature by Carr in [8] as an approximation methodology for the pricing of American style options. Within the MoL framework, time derivatives are replaced by finite differences, while derivatives with respect to stock price are kept intact. Consequently, the heat equation is reduced to an inhomogeneous ordinary differential-difference equation which can be solved analytically if the ODE is solvable, or numerically if no explicit solutions are available. We demonstrate that a modified version of the MoL, which incorporates a scaling of the stock price from one time-step to the next, applied to a Black-Scholes type PDE *exactly* replicates the prices of plain-vanilla options in which the underlying follows a VG process. MoL pricing is also exact for path-dependent options with payoff contingent on information on the lines only.

In the following, we review the financial interpretation of the MoL in terms of randomized maturity options as developed by Carr [9], introduce a version of the MoL which is appropriate for VG models and discuss barrier and Bermudan options. Conclusions and an appendix with mathematical derivations complete the paper.

2 Jumps and the Method of Lines

The MoL is an approximation scheme for solving quite general partial differential equations (PDE). In contrast to lattice pricing models, in the MoL only the time variable is discretized, while stock prices are continuous. This partial discretization leads to a chain of ordinary differential equations along the lines to be solved backwards from maturity. The MoL has been found to be an efficient numerical scheme for solving free boundary value problems, such as the pricing of American style option contracts [8]. A semi-explicit solution to the sequence of ordinary differential equations (ODE) arising from the MoL conversion of the Black-Sholes PDE was given in [10]. Carr gave an intriguing financial interpretation of the MoL in [9] which led us to the extension of the MoL to jump processes.

In the MoL pricing scheme for the Black-Scholes model, the price $P^{(n)}(S,K)$ at time $t = n\Delta t + t_0$ for $n = 0,\ldots,N$ of a European option maturing at $T = N\Delta t + t_0$ is given by

$$\left\{\frac{\sigma^2}{2}S^2 d_{SS} + rSd_S - r\right\} P^{(n)}(S,K) = \frac{1}{\Delta t}\left(P^{(n)}(S,K) - P^{(n+1)}(S,K)\right) \tag{1}$$

subject to the appropriate terminal and boundary conditions. For example, the conditions for a put option are as follows,

$$P^{(N)}(S,K) = (K-S)_+; \qquad \lim_{S\to\infty} P^{(n)}(S,K) = 0; \qquad \lim_{S\to 0} P^{(n)}(S,K) = e^{-rn\Delta t}K \tag{2}$$

As usual $(a)_+$ is equal to a if $a > 0$ and zero otherwise. Carr [9] notices that the MoL prices are *exact* for claims of random maturity τ. The stopping time τ is Erlang distributed with density function

$$\mathcal{P}\left(\tau \in (g, g+dg)\right) = \frac{g^{N-1}e^{-g/\Delta T}}{(N-1)!\Delta t^N}dg \tag{3}$$

In the limit as $\Delta t \to 0$ the above density reduces to a delta function, the expiration of the option is then certain to occur at time T, and the MoL price becomes exact. Notice that (3) can be viewed as the density for a Gamma process evaluated at time $N\Delta T$ with variance rate $\nu = \Delta t$ and mean rate 1.

Instead of considering random maturity claims, we can equivalently consider claims with fixed maturity subject to a stochastic time change. Stochastic time changes are related to jump processes. Since the Erlang distribution is a particular case of the Gamma distribution, the jump process underlying MoL pricing models is the VG process. To make the connection between VG and MoL pricing models precise, one has to specify a MoL methodology whereby stock prices drift at the risk-neutral rate.

The stock price, in the risk neutral measure, follows a VG process, given by

$$S_t = S_{t_0}\exp\left\{(r+\omega)(t-t_0) + X(\Gamma(t-t_0;1,\Delta t);\theta,\sigma)\right\}. \tag{4}$$

Here, $X(\tau;a,b)$ denotes a Brownian process evaluated at time τ with drift a and volatility b; $\Gamma(\tau;\mu,\nu)$ is a Gamma process evaluated at time τ with mean rate μ and variance rate ν; and ω is chosen so that the discounted stock price is a martingale (see for instance [6]), i.e. risk-neutrality is satisfied,

$$\mathbb{E}[S_t] = e^{(t-t_0)r}S_{t_0} \qquad \Rightarrow \qquad \omega = \frac{1}{\Delta t}\ln\left(1 - \left(\theta + \frac{1}{2}\sigma^2\right)\Delta t\right). \tag{5}$$

The following modified version of the MoL model, prices European options, with pay-off $f(S_{t+N\Delta t})$, maturing at time $T = t + N\Delta t$, *exactly* at times $t + n\Delta t$,

$$D_S P^{(n)}(S) = \frac{1}{\Delta t}\left(P^{(n)}(S) - e^{-r\Delta t} P^{(n+1)}\left(e^{(\omega+r)\Delta t} S\right)\right) \tag{6}$$

$$P^{(N)}(S) = f(S) \tag{7}$$

Appropriate boundary conditions are required on each slice, and the operator D_S is defined as follows,

$$D_S \equiv \frac{1}{2}\sigma^2 S^2 d_{SS} + \left(\theta + \frac{1}{2}\sigma^2\right) S d_S. \tag{8}$$

There are several important features of this modified MoL that should be elaborated on. Firstly, the drift which appears in (8) is not the risk-free rate. This is because in financial time the stock drifts according to θ rather than r. Secondly, in the operator D_S there is no constant term, i.e. the term $-rP$ in the usual Black-Sholes equation is missing. On reflection it is clear that such a term must be absent because the discounting occurs in real time and not financial time. Thirdly, once the price on one line is known, the price on the previous line is determined from an option with a scaled spot and discounted price. The discounting of the price is natural, and can be thought of as the spot price of the next line. The scaling of the spot itself can be understood from the fact that although the drift of the stock in financial time is not equal to the risk free rate, risk neutrality must still be enforced, hence across each line additional drifting must be imposed.

Volatility smiles for a VG process with $\sigma = 15\%$, $\theta = -20\%$, and $\Delta t = 1$ week are plotted in figure 1. We have calculated the relative error between the implied volatilities obtained using the exact prices in [6] and those obtained using the modified MoL. The largest relative error for the smiles in figure 1 was found to be $\sim 10^{-3}\%$ while the average relative error over the smiles was found to be $\sim 10^{-5}\%$. There was little difference in computation time between the two pricing schemes.

Within the MoL framework, information between time-steps is unavailable; consequently, any option which is contingent on that information, such as American, cannot be priced exactly using this methodology. However, Bermudan options, that can be exercised at times $n\Delta t$, and barrier options, checked with the same frequency, can be priced exactly as explained in the appendix. If the actual option is written so that the optionality is checked at closing prices, this approximation is not worse than the continuous time approximation. Furthermore, these algorithms allow for very efficient implementations, whereby errors are independent of the spacing between lines.

3 Extrapolation to a Wider Model Class

Although the time-step dictates the ν parameter in the VG model, it is possible to use the modified MoL to obtain approximate prices to VG models in which ν is different from Δt. Just as Carr [9] demonstrated that Richardson extrapolation to $\nu = 0$ reproduced the Black-Scholes value in the usual MoL framework, we propose to use an extrapolation scheme to obtain the prices of options for $\nu \neq \Delta t$. In figure 2 we have plotted the exact implied volatilities for one-month European options with various strike levels as a function of ν. Quadratic polynomials in $\ln \nu$ were fitted to the first three points at $\nu = \Delta t = 1, 2$ and 4 weeks, and then used to extrapolate the fourth point at $\nu = 8$ weeks. The fitting curves, also plotted in figure 2, demonstrate a very reliable prediction. The absolute error in predicting the implied volatility at the money was found to be negligible. The worst case turned out to be that of the out-of-the money put struck at 80% of the spot, for which the absolute error in implied volatility is 0.09%.

Just as in the case of plain-vanilla options, it is possible to use extrapolation to obtain the prices when $\nu \neq \Delta t$ in the exotic case. In figure 3 the boundary of a Bermudan option which can be exercised every 8 weeks is plotted as a function of $\nu = \Delta t = 1, 2, 4$ and 8 weeks. The crosses in figure 3 form the predicted 8 week boundary obtained by extrapolation from the first three points. The extrapolation is shown in figure 4 and is based on a fit to a quadratic polynomial of $\ln \nu$ to the first three boundaries. As both diagrams demonstrate, the errors obtained by this extrapolation method are minimal with a maximum absolute error of \$0.18 for the longest maturity option. The at-the-money prices and deltas are displayed in figures 5 and 6 respectively. They both are fitted to a linear function of ν. The error in the case of prices are negligible with the longest maturity option being underpriced by \$0.02. The maximum relative error for deltas is 0.7%. Given that the at the money price and delta, and the location of the exercise boundary, are known from the above extrapolation, the price function can be fully restored.

The discussed exercises demonstrate that extrapolation allows for the pricing of any VG model using the modified MoL framework, albeit the prices thus obtained are approximate.

4 Conclusions

We propose a generalization of the method of lines giving exact prices for the variance gamma process. The method admits a suggestive financial interpretation, applies to a large, albeit not exhaustive, family of VG models and is able to reproduces a large variety of implied volatility shapes. For this class of VG models, we are able to price exactly in terms of the solutions to ordinary differential equations, any payoff contingent only on price levels at the lines, including Bermudan and barrier options. We also illustrate that extrapolation techniques can be used to extend the applicability of our method. The numerical algorithms are very efficient and simple to implement.

5 Acknowledgements

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6 Appendix

Consider a stock which follows the VG process given in equation (4). Then any plain vanilla option on that stock can be priced by first conditioning on knowing the financial time given by the random time change, and then integrating over all financial times with the appropriate density. Suppose the pay-off of the option at the next time step is given by $f(S_{t+\Delta t})$, and denote the conditioned price of the option at current calendar time by $p(S_t, g)$, and the unconditioned price by $P(S_t)$. Then,

$$P(S_t) = \int_0^\infty dg \, \frac{e^{-g/\Delta t}}{\Delta t} \, p(S_t, g) \tag{9}$$

and

$$p(S_t, g) = e^{-r\Delta t}\mathbb{E}[f\,(S_{t+\Delta t})\,|g] \;=\; e^{-r\Delta t}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\sigma^2 g}}\, e^{-\frac{(x-\theta g)^2}{2\sigma^2 g}} f(e^{(\omega+r)\Delta t+x}S_t) \tag{10}$$

$$= e^{-r\Delta t+(\theta+\frac{1}{2}\sigma^2)g} P_{BS}\left(e^{(\omega+r)\Delta t}S_t, g, \left(\theta+\frac{1}{2}\sigma^2\right), \sigma\right) \tag{11}$$

where $P_{BS}(S, g, r, \sigma)$ denotes the Black-Sholes price of the plain-vanilla maturing at time g, S denotes the spot, r the risk-free rate and σ the volatility. Since, the Black-Sholes price satifies a well-known PDE, one can obtain the PDE that the conditioned price satisfies,

$$(-\partial_g + D_S)\, p(S_t, g) = 0 \tag{12}$$

where D_S is defined in (8), and the appropriate boundary conditions should be imposed. Applying the integral kernel appearing in the right hand side of (9) to the above PDE one finds,

$$-\frac{1}{\Delta t}\left[P(S_t) - \lim_{g\to 0} p(S_t, g)\right] + D_S P(S_t) = 0 \tag{13}$$

The limit can be computed by using (11) to obtain,

$$-\frac{1}{\Delta t}\left[P(S_t) - e^{-r\Delta t} f(e^{(\omega+r)\Delta t}S_t)\right] + D_S P(S_t) = 0 \tag{14}$$

This procedure can then be applied recursively to obtain the price of any pay-off that matures in multiples of Δt, and is given succinctly by the differential-difference equations (6) and (7).

It is trivial to generalize the procedure to price path dependent options which are contingent on information on the lines. Let $\tilde{P}(S_t)$ denote the price of the plain-vanilla option given by equation (14). The price of the exotic, $P(S_t)$, at the current time step, with rebate $R(S_t)$ and upper exercise level $S_{E.B.}$ will then be,

$$P(S_t) = \begin{cases} \tilde{P}(S_t) & , S_t < S_{E.B.} \\ R(S_t) & , S_t \geq S_{E.B.} \end{cases} \tag{15}$$

An up and out Barrier is obtained by taking $S_{E.B.} = H$ the upper barrier, and the rebate $R(S_t) = R$ the rebate amount. A Bermudan put can be priced by choosing $S_{E.B.}$ such that $\tilde{P}(S_{E.B.}) = K - S_{E.B.}$, and rebate $R(S_t) = (K - S_{E.B.})_+$. Generalizations to double barriers and other exotics are fairly straightforward. Once the price of the exotic is obtained on the current line, that price is treated as the pay-off to obtain the exotic price on the previous line and so on.

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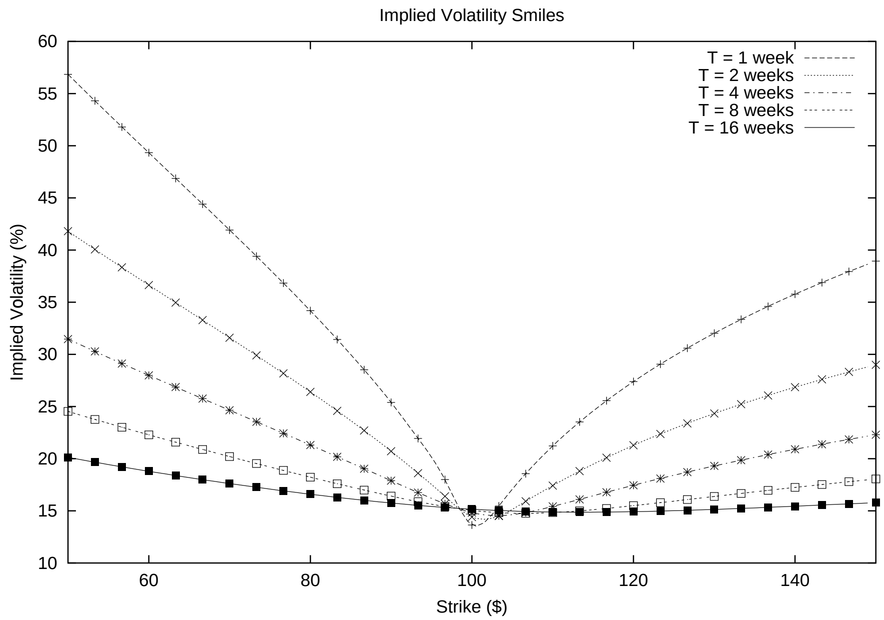

Figure 1: The implied smiles of the VG model for maturities of 1, 2, 4, 8 and 16 weeks with parameters: $\nu = \Delta t = 1$ week, $\sigma = 15\%$, $\theta = -20\%$ and $r = 5\%$

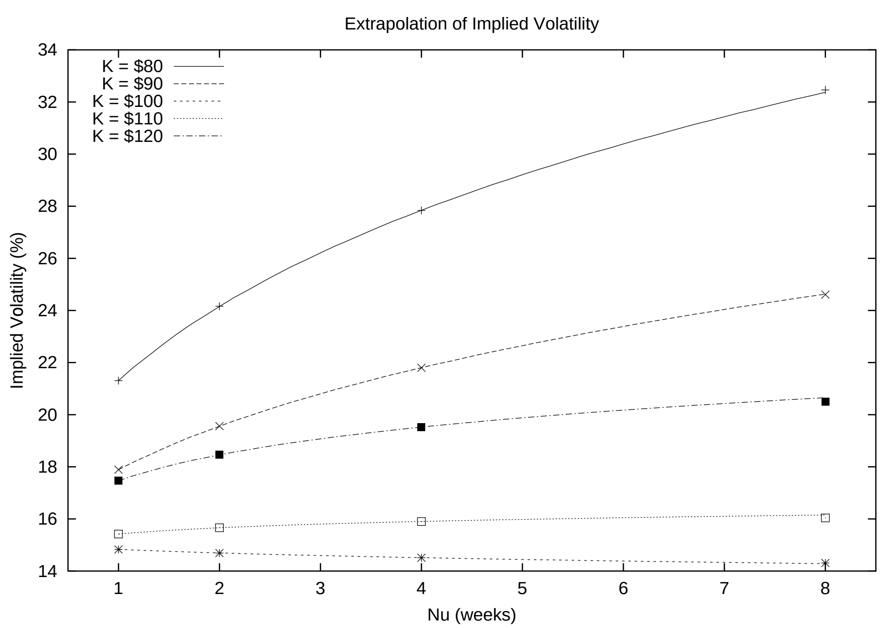

Figure 2: Extrapolation of the implied volatility of one month European options for a VG model with $\nu = 8$ weeks using a fit to volatilities obtained with $\nu = \Delta t = 1, 2$ and 4 weeks. The model parameters were $\sigma = 15\%$, $\theta = -20\%$ and $r = 5\%$ and the spot was taken to be \$100

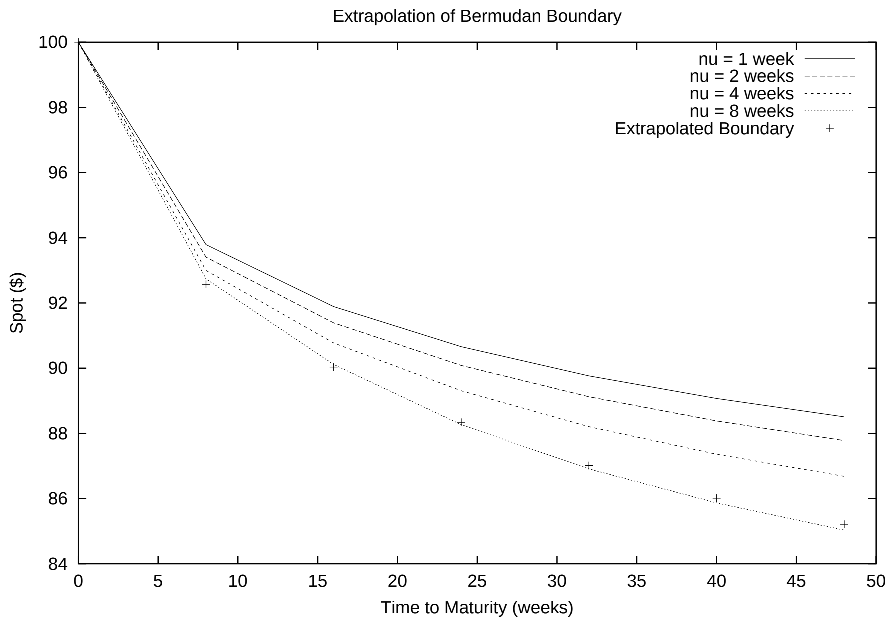

Figure 3: The boundary of a Bermudan option which can be exercised every 8 weeks plotted for several values of $\nu = \Delta t$. The crosses show the boundary with $\nu = 8$ weeks obtained by extrapolation using the first three boundaries. The model parameters were $\sigma = 15\%$, $\theta = -20\%$ and $r = 5\%$ and the spot was taken to be \$100

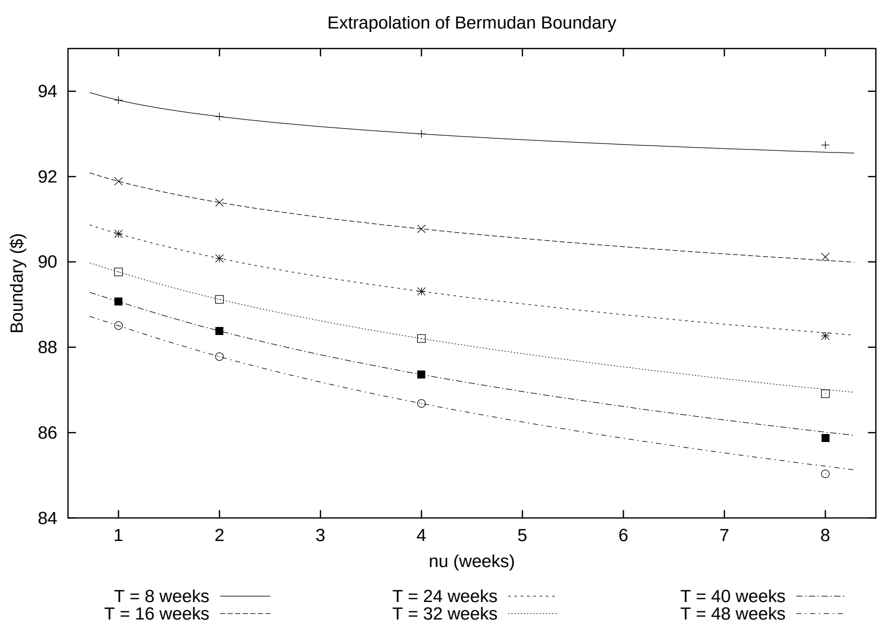

Figure 4: Extrapolation of the boundary points shown in figure 3 for each maturity

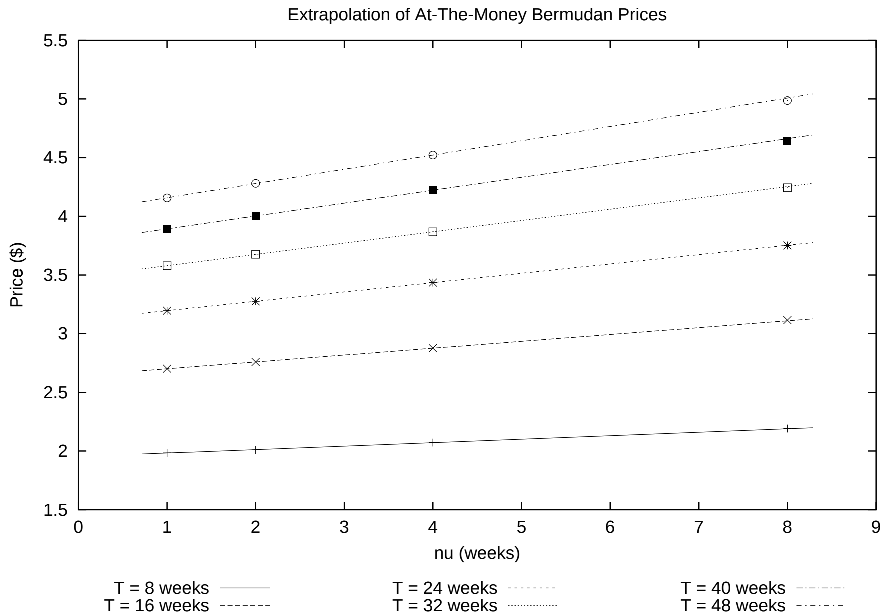

Figure 5: The prices of the at-the-money options whose boundaries are shown in figure 3. The lines indicate a fit to the first three prices extrapolated to the fourth

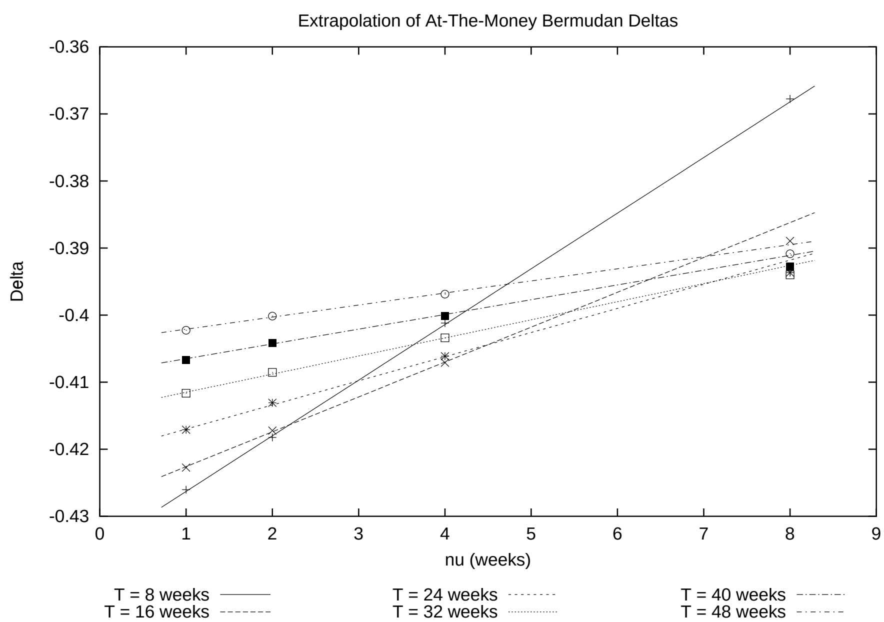

Figure 6: The deltas of the at-the-money options whose boundaries are shown in figure 3. The lines indicate a fit to the first three deltas extrapolated to the fourth