Hedging Complex Barrier Options

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Summary Page

Abstract

We show how several complex barrier options can be hedged using a portfolio of standard European options. These hedging strategies only involve trading at a few times during the option's life. Since rolling, ratchet, and lookback options can be decomposed into a portfolio of barrier options, our hedging results also apply to them.

Hedging Complex Barrier Options

1 Introduction

Barrier options have become increasingly popular in many over-the-counter markets[1]. This popularity is due to the additional flexibility which barrier options confer upon their holders. In general, barrier options allow investors to revise their vanilla option positions costlessly at the first time one or more critical price levels are reached. For example, an out-option with a constant rebate allows an investor to effectively sell the vanilla option at a fixed price at the first hitting time of the barrier. Similarly, an in-option allows an investor to effectively buy a vanilla option at the first hitting time at no cost beyond the initial premium. Used in combination, portfolios of barrier options can change the strike or maturity of a vanilla option at the first hitting time of one or more critical price levels.

The seminal paper by Merton[17] values a down-and-out call option in closed form. The valuation relies on the ability to perfectly replicate the payoffs to the barrier option using a dynamic strategy in the underlying asset. A series of papers (see [4, 6, 7, 8, 9]) shows how to alternatively replicate the payoffs of barrier options using a static position in vanilla options. Since the economic assumptions underlying the static replication are the same as for the dynamic one, the resulting static valuation matches the dynamic one. However, the efficacy of the hedge in static models is likely to be more robust to violations of standard assumptions such as continuously open markets, constant volatility, and no transactions costs.

The purpose of this paper is to extend these static hedging results for single barrier options to more complex barrier options. In particular, we will examine the following types of barrier options:

1. **Partial Barrier Options:** For these options, the barrier is active only during an initial period. In other words, the barrier disappears at a prescribed time. In general, the payoff at maturity may be a function of the spot price at the time the barrier disappears.

2. **Forward Starting Barrier Options:** For these options, the barrier is active only over the latter period of the option's life. The barrier level may be fixed initially, or alternatively, may be set at the forward start date to be a specified function of the contemporaneous spot price. The payoff may again be a function of the spot price at the time the barrier becomes active.

[1] For a description of exotic options in general and barrier options in particular, see Nelken[18] and Zhang[20].

3. **Double Barrier Options:** Options that knock in or out at the first hitting time of either a lower or upper barrier.

4. **Rolling Options:** These options are issued with a sequence of barriers, either all below (for roll-down calls) or all above (for roll-up puts) the initial spot price. Upon reaching each barrier, the option strike is lowered (for calls) or raised (for puts). The option is knocked out at the last barrier.

5. **Ratchet Options:** These options differ from rolling options in only two ways. First, the strike ratchets to the barrier each time a barrier is crossed. Second, the option is not knocked out at the last barrier. Instead, the strike is ratcheted for the last time.

6. **Lookback Options:** The payoff of these options depends upon the maximum or the minimum of the realized price over the lookback period. The lookback period may start before or after the valuation date but must end at or before the option's maturity.

As shown in [4] and [7], the last three categories above may be decomposed into a sum of single barrier options. Consequently, rolling, ratchet, and lookback options can be staticly hedged using the results of the foregoing papers. Furthermore, the decomposition into barrier options is model-independent. Thus, as new static hedging results for single barrier options are developed, these results will automatically hold for these multiple barrier options.

The structure of this paper is as follows. The next section reviews previous results on static hedging. The next six sections examine the static replication of the six types of claims described above. The last section reviews the paper. Three appendices contain technical results.

2 Review of Static Hedging

2.1 Static Hedging of Path-Independent Securities

Breeden and Litzenberger[5] showed that any path-independent payoff can be achieved by a portfolio of European calls and puts. In particular, Carr and Chou[6] showed that any twice differentiable[2] payoff $f(S)$ can be written as:

$$f(S) = f(F_0)+(S-F_0)f'(F_0)+\int_0^{F_0} f''(K)(K-S)^+ dK+\int_{F_0}^{\infty} f''(K)(S-K)^+ dK. \quad (1)$$

where F_0 can be any fixed constant, but will henceforth denote the initial forward price. Thus, any such payoff can be uniquely decomposed into the payoff

[2] If $f(\cdot)$ is not twice differentiable, then (1) still holds if $f')(\cdot)$ and $f''(\cdot)$ are interpreted as generalized functions such as Heaviside step functions and Dirac delta functions.

from a static position in $f(F_0)$ unit discount bonds, $f'(F_0)$ initially costless forward contracts[3], and the continuum of initially out-of-the-money options. To prevent arbitrage, the initial value V of the payoff must be the cost of this replicating portfolio:

$$V = f(F_0)B_0 + \int_0^{F_0} f''(K)P(K)dK + \int_{F_0}^{\infty} f''(K)C(K)dK, \tag{2}$$

where B_0 is the initial value of the unit bond, and $P(K), K \leq F_0$ and $C(K), K \geq F_0$ are the initial values of out-of-the-money forward puts and calls respectively. Note that the second term in (1) does not appear in (2) since the forward contracts held are initially costless. In what follows, we will be providing path-independent payoffs which lead to values matching those of path-dependent payoffs. We will leave it to the reader to use (1) to recover the static replicating portfolio and to use (2) to recover its value.

2.2 Static Hedging of Single Barrier Claims

A single barrier claim is one that provides a specified payoff at maturity so long as a barrier for the underlying price has been hit (in-claim) or has not been hit (out-claim). This subsection shows that one can replicate the payoff of any single barrier claim with a portfolio of vanilla European options. The portfolio is *static* in the sense that we never need to trade unless the claim expires or its underlying asset hits a barrier.

Our static hedging results all rely on Lemma 1 in Carr and Chou[6], which is repeated below and proven in Appendix 2:

Lemma 1 *In a Black-Scholes economy, suppose that X is a portfolio of European options expiring at time T with payoff:*

$$X(S_T) = \begin{cases} f(S_T) & \text{if } S_T \in (A, B), \\ 0 & \text{otherwise.} \end{cases}$$

For $H > 0$, let Y be a portfolio of European options with maturity T and payoff:

$$Y(S_T) = \begin{cases} \left(\frac{S_T}{H}\right)^p f(H^2/S_T) & \text{if } S_T \in (H^2/B, H^2/A), \\ 0 & \text{otherwise} \end{cases}$$

where the power $p \equiv 1 - \frac{2(r-d)}{\sigma^2}$ and r, d, σ are the interest rate, dividend rate and volatility rate respectively.

Then, X and Y have the same value whenever the spot equals H.

[3] Note that since bonds and forward contracts can themselves be created out of options, the spectrum of options is sufficiently rich so as to allow the creation of any sufficiently smooth payoff, as shown in Breeden and Litzenberger[5].

The payoff of Y is the reflection[4] of the payoff of X along axis H. Note that A or B can be assigned to be 0 or ∞ respectively. This lemma is model-dependent in that it uses the Black-Scholes assumptions.

The lemma can be used to find the replicating portfolio of any single barrier claim. For example, consider a down-and-in claim which pays $f(S_T)$ at T provided a lower barrier H has been hit over $[0, T]$. From the previous section, we know that a portfolio of vanilla options can be created which provides an *adjusted payoff*, defined as:

$$\hat{f}(S_T) \equiv \begin{cases} 0 & \text{if } S_T > H, \\ f(S_T) + \left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H. \end{cases} \tag{3}$$

If the lower barrier is never hit, then the vanilla options expire worthless, matching the payoff of zero from the down-and-in. If the barrier is hit over $[0, T]$, then Lemma 1 indicates that at the first hitting time, the value of the $\left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right)$ term matches the value of a payoff $f(S_T)1_{S_T>H}$, where 1_E denotes an indicator function of the event E. Thus, the options providing the payoff $\left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right)$ can be sold off with the proceeds used to buy options delivering the payoff $f(S_T)1_{S_T>H}$. Consequently, after rebalancing at the hitting time, the total portfolio of options delivers a payoff of $f(S_T)$ as required. It follows that whether the barrier is hit or not, the portfolio of European options providing the adjusted payoff f replicates the payoffs of the down-and-in claim.

By in-out parity[5], the adjusted payoff corresponding to a down-and-out claim is:

$$\hat{f}(S_T) \equiv \begin{cases} f(S_T) & \text{if } S_T > H, \\ -\left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H. \end{cases} \tag{4}$$

The reflection principle implicit in Lemma 1 can also be applied to up-barrier claims. The adjusted payoff corresponding to an up-and-in security is:

$$\hat{f}(S_T) \equiv \begin{cases} f(S_T) + \left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T > H, \\ 0 & \text{if } S_T < H. \end{cases} \tag{5}$$

Similarly, an up-and-out security is accociated with the adjusted payoff:

$$\hat{f}(S_T) \equiv \begin{cases} -\left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T > H, \\ f(S_T) & \text{if } S_T < H. \end{cases} \tag{6}$$

Note that all of the above adjusted payoffs can be obtained in a simple manner if one already has a pricing formula, either from the literature or from

[4] The reflection is geometric and accounts for drift.

[5] In-out parity is a relationship which states that thepayoffs and values of an in-claim and an out-claim sum to the payoffs and values of an unrestricted claim.

dynamic replication arguments. In this case, the adjusted payoff is the limit of the pricing formula $V(S_T, \tau)$ as the time to maturity approaches zero (after removing domain restrictions such as $S > H$).

$$\hat{f}(S_T) = \lim_{\tau \downarrow 0} V(S_T, \tau), \qquad S_T > 0.$$

3 Partial Barriers

A partial barrier option[6] has a barrier that is active only during part of the option's life. Typically, the barrier is active initially, and then disappears at some point during the option's life. One could also imagine the opposite situation, where the barrier starts inactive and becomes active at some point. We denote these options as *forward-starting options* and discuss them in section 4.

We will present two different hedging strategies in this section. In the first method, we will rebalance when the barrier disappears. This method is very general, in that the final payoff of the option can depend upon the spot price at the time the barrier disappears. Usually, the payoff is not a function of this price and depends only on the final spot price. In this case, we can apply a second hedging method, which is superior to the first method in that it does not require rebalancing when the barrier disappears.

We will examine down-barriers and leave it to the reader to develop the analogous results for up-barriers. Consider a partial barrier option with maturity T_2, which knocks out at barrier H. Let $T_1 \in (0, T_2)$ denote the time when the barrier expires. At time T_1, either the option has knocked out or else it becomes a European claim with some payoff at time T_2. This payoff may depend upon the spot price at time T_1, which we denote by S_1. Using risk-neutral valuation (see Appendix 1), we can always find the function $V(S_1)$ relating the value at T_1 of this payoff to S_1.

Define the adjusted payoff at time T_1 as:

$$\hat{f}(S_1) = \begin{cases} V(S_1) & \text{if } S_1 > H, \\ -\left(\frac{S_1}{H}\right)^p V(H^2/S_1) & \text{if } S_1 \leq H. \end{cases}$$

Thus, our replicating strategy is as follows:

1. At initiation, purchase a portfolio of European options that gives the adjusted payoff $\hat{f}(S_1)$ at maturity date T_1.

2. If the barrier is reached before time T_1, liquidate the portfolio. From Lemma 1, the portfolio is worth zero.

3. At time T_1, if the barrier has not been reached, use the payoff from the expiring options to purchase the appropriate portfolio of European options maturing at time T_2.

[6] Heynen and Kat[13] provide analytic valuation formulas for partial barrier options.

We can also find a replicating strategy for an in-barrier claim. Consider an exotic claim maturing at T_2 with no barrier, but with a payoff that depends upon S_1. Standard techniques such as risk-neutral valuation allow us to identify the function $V(S_1)$ giving the value at time T_1. Therefore, by in-out parity, the adjusted payoff at time T_1 is:

$$\hat{f}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ V(S_1) + \left(\frac{S_1}{H}\right)^p V(H^2/S_1) & \text{if } S_1 \leq H. \end{cases}$$

Our replicating strategy is as follows:

1. At initiation, purchase a portfolio of European options that pays off $\hat{f}(S_1)$ at time T_1.
2. If the barrier is reached before time T_1, then rebalance the portfolio to have payoff $V(S_1)$ at time T_1 *for all* S_1. By single barrier techniques, the value of the adjusted payoff term $\left(\frac{S_1}{H}\right)^p V(H^2/S_1)$ exactly matches the value of the payoff $V(S_1)1_{S_1>H}$.
3. At time T_1, if the barrier has not been reached, our payoff is zero.

For this hedging method, the possible rebalancing points are the first passage time to the barrier and time T_1. We now present a second method that only requires rebalancing at the first passage time. However, we require the payoff at time T_2 to be independent of S_1. Our replicating portfolio will use options that expire at both T_1 and T_2.

Let the partial barrier claim payoff at time T_2 be $g(S_2)$, where S_2 is the spot price at time T_2. From the last section, we know that we can form a portfolio of European options with payoff $g(S_2)$ at time T_2. Let $V(S_1)$ denote the function relating its value at time T_1 to S_1. Now, suppose our partial barrier claim is a down-and-out. Then, the desired payoff at time T_1 is:

$$f(S_1) = \begin{cases} V(S_1) & \text{if } S_1 > H, \\ -\left(\frac{S_1}{H}\right)^p V(H^2/S_1) & \text{if } S_1 \leq H. \end{cases}$$

Unfortunately, our current portfolio of options maturing at T_2 has a value at T_1 of only $V(S_1)$. Thus, we must add a portfolio of European options maturing at T_1 to make up this difference. These options provide the following adjusted payoff at time T_1:

$$\hat{f}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ -V(S_1) - \left(\frac{S_1}{H}\right)^p V(H^2/S_1) & \text{if } S_1 \leq H. \end{cases}$$

Our replicating strategy is as follows:

1. At initiation, purchase a portfolio of European options that:
   - provides payoff $g(S_2)$ at maturity T_2, and

- provides payoff $\hat{f}(S_1)$ at maturity T_1.

2. Upon reaching the barrier before time T_1, liquidate all options. From Lemma 1, our portfolio will be worth zero.

3. If the barrier is not reached before time T_1, our payoff will be $g(S_2)$ at time T_2 as desired. Note that it is impossible for the options maturing at time T_1 to pay off without the barrier being reached.

Interestingly, the options maturing at T_1 never finish in-the-money. If the barrier is reached, they are liquidated. Otherwise, they expire out-of-the-money at time T_1. Thus, our only rebalancing point is the first passage time to the barrier.

For a down-and-in claim, we can apply in-out parity. Our replicating portfolio is simply a portfolio of European options that pays off $-\hat{f}(S_1)$ at maturity date T_1. If the barrier is not hit by T_1, these options expire worthless as desired. If the barrier is hit before T_1, the value of this portfolio matches the value of a portfolio of European options paying off $g(S_2)$ at time T_2. Thus, the options maturing at T_1 can be sold off with the proceeds used to buy the options maturing at T_2. Using this second method, one only needs to rebalance at the first passage time to the barrier, if any.

To illustrate both methods, consider a down-and-out partial barrier call with strike K, maturity T_2, partial barrier H, and barrier expiration T_1. Using the first hedging method, our initial replicating portfolio will have maturity T_1 and payoff (see Figure 1):

$$\hat{f}(S_1) = \begin{cases} C(S_1) & \text{if } S_1 > H, \\ -\left(\frac{S_1}{H}\right)^p C\left(\frac{H^2}{S_1}\right) & \text{if } S_1 < H, \end{cases} \tag{7}$$

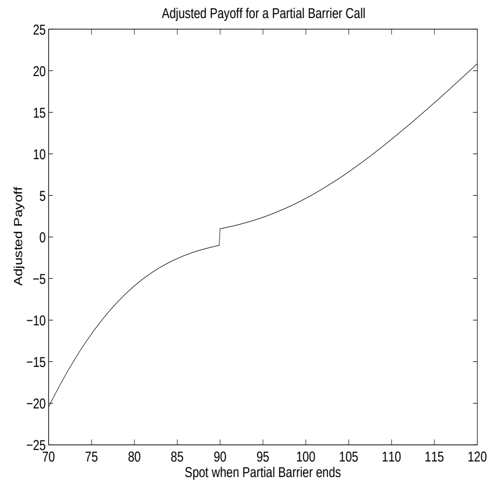
where $C(S_1)$ is the Black-Scholes call pricing formula for a call with spot S_1, strike K, and time to maturity $T_2 - T_1$. The initial value of the partial barrier call is just the discounted expected value[7] of $\hat{f}$ at time T_1.

The payoff of this option is independent of S_1, so we can also apply the second hedging method. The portfolio of options maturing at T_2 reduces to a single call struck at K. The portfolio of options maturing at T_1 has the payoff (see Figure 2):

$$\hat{f}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ -C(S_1) - \left(\frac{S_1}{H}\right)^p C(H^2/S_1) & \text{if } S_1 \leq H. \end{cases}$$

The initial value of the barrier option is given by the sum of the initial values of the options maturing at T_1 and T_2.

[7] See Appendix 3 for a closed form solution for this value.



$$(r = 0.05,\ d = 0.03,\ \sigma = .15,\ H = 90,\ K = 100,\ T_2 - T_1 = .5)$$

Figure 1: Adjusted payoff for a Partial Barrier Call Using First Hedging Method.

4 Forward Starting Barrier Options

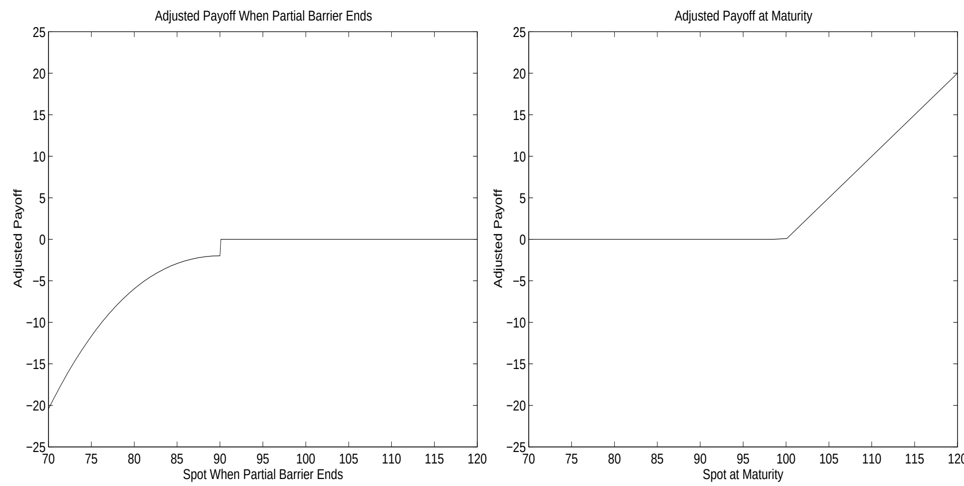
For forward-starting options, the barrier is active only over the latter period of the option's life. As we shall see, forward-starting barrier options are very similar to partial barrier options.

Again, we will present two different replicating methodologies. The first method is more general and can be applied to cases where the barrier and/or payoff depend upon the spot price when the barrier becomes active. This method possibly requires rebalancing when the barrier appears and at the first passage time to the barrier. The second method requires that the barrier and payoff be independent of the spot price when the barrier appears, but only requires rebalancing at most once.

Consider a forward-starting claim maturing at T_2, and let the barrier appear at time T_1. At time T_1, the exotic becomes identical to a single barrier claim. Using the static hedging techniques described in subsection 2.2, we can price[8] the exotic at time T_1 as $V(S_1)$.

Our first hedging method is create a portfolio of European options that pays off $V(S_1)$ at time T_1. At time T_1, the payoff from these options will be used to buy a portfolio of options maturing at T_2 which replicates the payoffs of a

[8] Note that at the forward start date, we also know how to price a partial barrier claim using the results of the last section. Consequently, one can use the techniques of this section to price a barrier claim, where the barrier is active only over a period occurring strictly within the claim's life.



$(r = 0.05,\ d = 0.03,\ \sigma = .15,\ H = 90,\ K = 100,\ T_2 - T_1 = .5)$

Figure 2: Adjusted payoffs for a Partial Barrier Call Using Second Hedging Method.

single barrier claim maturing at T_2. Thus, our hedging strategy always requires rebalancing at time T_1. The subsequent single barrier replication may require an additional rebalancing.

An important special case arises if $V(S_1)$ may be written as $S_1 \times n(\cdot)$, where $n(\cdot)$ is independent of S_1. This situation arises for barrier options where the strike and barrier are both proportional to S_1. In this case, the hedge is to buy $n(\cdot)e^{-dT_1}$ shares at time 0 and re-invest dividends until T_1. The shares are then sold and the proceeds are used to buy options providing the appropriate adjusted payoff at T_2.

We now discuss the second method, which is applicable when the barrier and payoff are independent of S_1. As before, we will examine down-barriers and leave it to the reader to apply the same techniques to up-barriers. Consider a forward-starting knockout with payoff $g(S_2)$ at time T_2 and barrier H active over $[T_1, T_2]$. At T_1, the security is identical to a single barrier claim, so we would like our adjusted payoff at time T_2 to be:

$$\hat{g}^{out}(S_2) = \begin{cases} g(S_2) & \text{if } S_2 > H, \\ -\left(\frac{S_2}{H}\right)^p g(H^2/S_2) & \text{if } S_2 \leq H. \end{cases}$$

Let $V(S_1)$ denote the value at T_1 of this adjusted payoff. To replicate the value of the forward starting claim, we need our portfolio at time T_1 to be worth:

$$f(S_1) = \begin{cases} V(S_1) & \text{if } S_1 > H, \\ 0 & \text{if } S_1 \leq H. \end{cases}$$

The payoff of zero below the barrier arises because our forward-starting option is defined to be worthless if the stock price is below the knockout barrier when the barrier is activated. Thus, we will add options maturing at time T_1 with payoff:

$$\hat{f}^{out}(S_1) = \begin{cases} 0 & \text{if } S_2 > H, \\ -V(S_1) & \text{if } S_2 \leq H. \end{cases}$$

Our replicating strategy is:

1. At initiation, purchase a portfolio of European options that:
   - provides payoff $\hat{g}^{out}(S_2)$ at maturity T_2, and
   - provides payoff $\hat{f}^{out}(S_1)$ at maturity T_1.
2. If the spot price at time T_1 is below H, the exotic has knocked out, so liquidate the portfolio.
3. Otherwise, the portfolio is held at T_1. If the barrier is hit between time T_1 and T_2, the portfolio is liquidated. Otherwise, the portfolio pays off $g(S_2)$ at T_2.

Note that whenever the portfolio is liquidated before maturity, it has zero value by construction.

For knock-in claims, one can apply in-out parity. Our replicating portfolio consists of options maturing at T_2 with payoff:

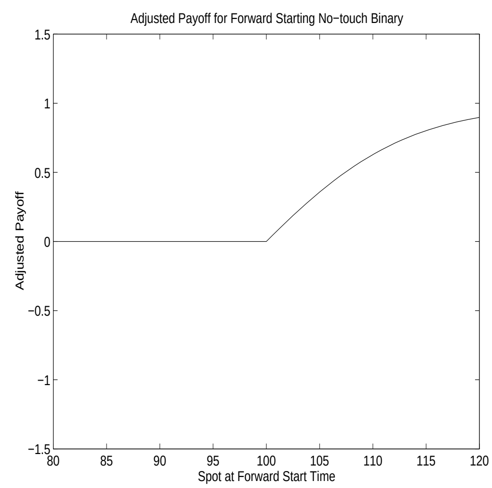
$$\hat{g}^{in}(S_2) = \begin{cases} 0 & \text{if } S_2 > H, \\ g(S_2) + \left(\frac{S_2}{H}\right)^p g(H^2/S_2) & \text{if } S_2 \leq H, \end{cases}$$

and options maturing at time T_1 with payoff:

$$\hat{f}^{in}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ V(S_1) & \text{if } S_1 \leq H, \end{cases}$$

where $V(S_1)$ was defined previously as the time T_1 value of the payoff $\hat{g}^{out}$ at time T_2.

To see why this portfolio replicates the payoffs of a forward-starting knockin claim, note that if $S_1 > H$ at time T_1, then the $\hat{f}^{in}(S_1)$ replicas expire worthless. The remaining options replicate the payoffs of a a single barrier knockin, as required. On the other hand, if $S_1 \leq H$ at time T_1, then the options maturing at T_2 have a value at T_1 equal to that of an in-barrier claim, while the options maturing at T_1 have a payoff at T_1 equal to the value of an out-barrier claim. By



$(r = 0.05,\ d = 0.03,\ \sigma = .15,\ H = 100,\ T_2 - T_1 = .5)$.

Figure 3: Adjusted payoff for Forward Starting No-touch Binary Using First Hedging Method.

in-out parity, the sum of the two values is that of a vanilla claim paying $\hat{g}(S_2)$ at T_2, as required. To maintain the hedge to T_2, the payoff from the options maturing at T_1 is used to buy the appropriate position in options maturing at T_2. Thus, in contrast to the first method, at most one rebalancing is required.

To illustrate both methods, consider a forward-starting no-touch binary[9] with down barrier H, maturity T_2, and barrier start date T_1. Using the first method, the portfolio of options with maturity T_1 has payoff (as shown in Figure 3):

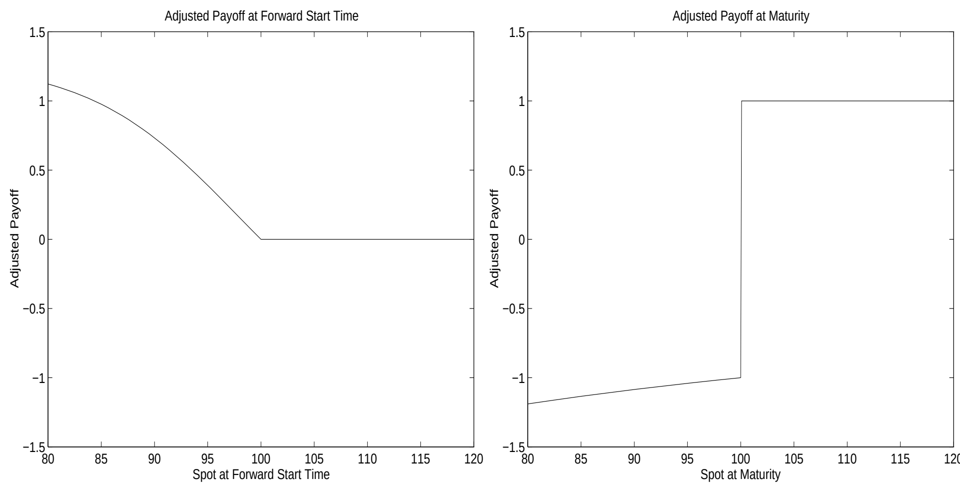
$$f(S_1) = \begin{cases} NTB(S_1) & \text{if } S_1 > H, \\ 0 & \text{if } S_1 < H, \end{cases}$$

where $NTB(S_1)$ is the Black-Scholes price of a no-touch binary[10] with spot S_1, time to maturity $T_2 - T_1$, and barrier H.

Since the barrier and payoff are independent of S_1, we can also apply the second method. The portfolio of options with maturity T_2 has payoff (see Figure 4):

$$\hat{g}^{out}(S_2) = \begin{cases} 1 & \text{if } S_2 > H, \\ -\left(\frac{S_2}{H}\right)^p & \text{if } S_2 \leq H, \end{cases}$$

[9] A no-touch binary pays one dollar at maturity if the barrier has not been hit.
[10] See Reiner and Rubinstein[19] for the formula.



$(r = 0.05,\ d = 0.03,\ \sigma = .15,\ H = 100,\ T_2 - T_1 = .5)$.

Figure 4: Adjusted payoffs for Forward Starting No-touch Binary Using Second Hedging Method.

and the portfolio of options with maturity T_1 has payoff:

$$\hat{f}^{out}(S_1) = \begin{cases} 0 & \text{if } S_1 > H, \\ -NTB(S_1) & \text{if } S_1 \leq H, \end{cases}$$

where we extend the $NTB(\cdot)$ formula to values below H.

5 Double Barriers

A double barrier claim is knocked in or out at the first passage time to either a lower or upper barrier. Double barrier calls and puts have been priced analytically in Kunitomo and Ikeda[16] and Beaglehole[1], and using Fourier series in Bhagavatula and Carr[2]. In analogy with the single barrier case, our goal is to find a portfolio of European options, so that at the earlier of the two first passage times and maturity, the value of the portfolio exactly replicates the payoffs of the double barrier claim. In order to do this, we will need to use multiple reflections.

Consider a double knockout with down barrier D, up barrier U, and maturity date T. We begin by dividing the interval $(0, \infty)$ into regions as in Figure 5.

We can succinctly define the regions as:

$$\text{Region } k = \left(\left(\frac{U}{D}\right)^k D, \left(\frac{U}{D}\right)^k U \right).$$

To specify the adjusted payoff for a region i, we will use the notation $\hat{f}_{(i)}(S_T)$. We begin with $\hat{f}_{(0)}(S_T) = f(S_T)$.

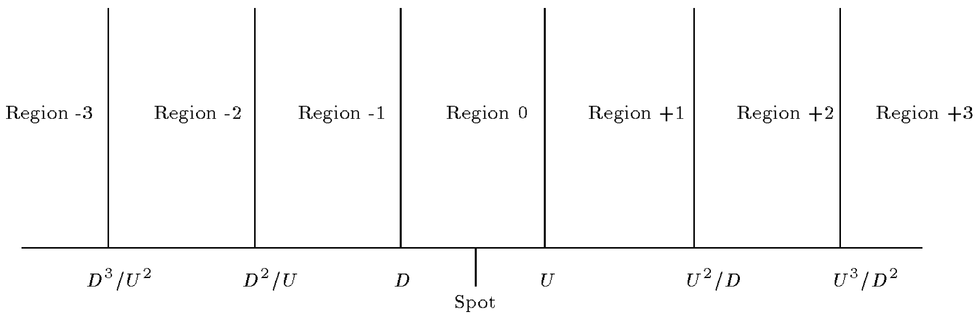


Figure 5: Dividing $(0, \infty)$ into regions.

From Lemma 1, we see that for a reflection along D, the region k (eg. $k = -2$) would be the reflection of region $-k - 1$ (eg. $-k - 1 = +1$). Similarly, for reflection along U, region k would be the reflection of region $-k + 1$.

It is useful to define the following two operators:

$$R_D(\hat{f}(S_T)) = -\left(\frac{S_T}{D}\right)^p \hat{f}(D^2/S_T) \text{ and } R_U(\hat{f}(S_T)) = -\left(\frac{S_T}{U}\right)^p \hat{f}(U^2/S_T).$$

It follows that:

$$\hat{f}_{(k)}(S_T) = R_D(\hat{f}_{(-k-1)}(S_T)), \qquad \text{for } k < 0$$

and

$$\hat{f}_{(k)}(S_T) = R_U(\hat{f}_{(-k+1)}(S_T)), \qquad \text{for } k > 0.$$

Note that R_U and R_D bijectively map between the corresponding regions. Also, we are taking the negative of the reflection, so that the valuation of the payoffs will cancel. By induction, we can completely determine the entire adjusted payoff as:

$$\hat{f}_{(k)}(S_T) = \begin{cases} f(S_T) & \text{for } k = 0, \\ \underbrace{R_D \circ R_U \circ R_D \ldots}_{k \text{ operators}}(f(S_T)) & \text{for } k < 0, \\ \underbrace{R_U \circ R_D \circ R_U \ldots}_{k \text{ operators}}(f(S_T)) & \text{for } k > 0. \end{cases}$$

A portfolio of European options that delivers the above adjusted payoff replicates the payoff to a double barrier claim. If we never touch either barrier, then the adjusted payoff from region 0 matches the payoff of the original exotic. Upon reaching a barrier, the values of the payoffs above the barrier are cancelled by the values of the payoffs below the barrier. Therefore, our portfolio is worth zero at either barrier at which point we can liquidate our position.

To find the adjusted payoff for a double knockin claim, we apply in-out parity. The adjusted payoff is given by:

$$\hat{f}_{(k)}(S_T) = \begin{cases} 0 & \text{for } k = 0, \\ f(S_T) - \underbrace{R_D \circ R_U \circ R_D \ldots}_{k \text{ operators}}(f(S_T)) & \text{for } k < 0, \\ f(S_T) - \underbrace{R_U \circ R_D \circ R_U \ldots}_{k \text{ operators}}(f(S_T)) & \text{for } k > 0. \end{cases}$$

As an example, consider a double knockout binary, which pays one dollar at maturity if neither barrier is hit beforehand. Then, $f(S_T) = 1$, and the adjusted payoff is (see Figure 6):

$$\hat{f}(S_T) = \begin{cases} -\left(\frac{S_T}{U}\right)^p \left(\frac{D}{U}\right)^{jp} & \text{in region } 2j+1, \\ \left(\frac{U}{D}\right)^{jp} & \text{in region } 2j, \end{cases}$$

where j is an integer and recall $p = 1 - \frac{2r-d}{\sigma^2}$. Two special cases are of interest. For $r = d$, we have $p = 1$, and the adjusted payoff become piecewise linear. For $r - d = \frac{1}{2}\sigma^2$, we have $p = 0$, and the adjusted payoff is piecewise constant.

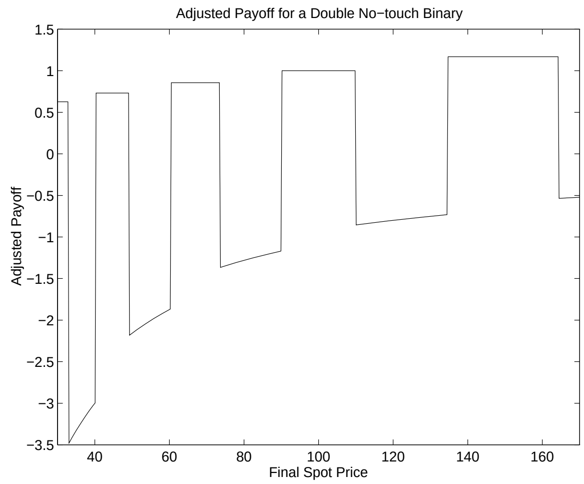
To value the double knocout binary, we simply value the adjusted payoff in each region and sum over all regions. If the current spot price is S, the value of the payoff in region k is:

$$V(S,k) = \begin{cases} -\left(\frac{S}{U}\right)^p \left(\frac{D}{U}\right)^{jp} e^{-rT} \left[N\left(\frac{\ln(x_1)-\mu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln(x_2)-\mu T}{\sigma\sqrt{T}}\right)\right] & \text{in region } k = 2j+1, \\ \left(\frac{U}{D}\right)^{jp} e^{-rT} \left[N\left(\frac{\ln(x_1)+\mu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln(x_2)+\mu T}{\sigma\sqrt{T}}\right)\right] & \text{in region } k = 2j, \end{cases}$$

where $x_1 = \frac{SD^{k-1}}{U^k}$, $x_2 = \frac{SD^k}{U^{k+1}}$, and $\mu = r - d - \frac{1}{2}\sigma^2$. Therefore, the value of the double knockout binary is the sum of the values for each region.

$$NTB(S) = \sum_{k=-\infty}^{\infty} V(S,k).$$

Although this sum is infinite, we can get an accurate price with only a few terms. Intuitively, the regions far removed from the barriers will contribute little to the price. Therefore, we only need to calculate the sum for a few values of k near 0. In Table 1, we illustrate this fact.



$(r = 0.05,\ d = 0.03,\ \sigma = .15,\ D = 90,\ U = 110)$

Figure 6: Adjusted payoff for Double No-touch Binary.

6 Rolling Options

The replication of rolldown calls[11], ratchet calls, and lookback calls was examined by Carr, Ellis, and Gupta[7]. In the next three sections, we review their decomposition into single barrier options and then apply our techniques for barrier option replication.

A rolldown call is issued with a series of barriers, $H_1 > H_2 > \ldots > H_n$, which are all below the initial spot. At initiation, the roll-down call resembles a European call with strike K_0. If the first barrier H_1 is hit, the strike is rolled down to a new strike $K_1 < K_0$. Upon hitting each subsequent barrier $H_i < H_{i-1}$, the strike is again rolled down to $K_i < K_{i-1}$. When the last barrier H_n is hit, the option knocks out.

A roll-down call can be decomposed in terms of knockout options as:

$$RDC = DOC(K_0, H_1) + \sum_{i=1}^{n-1} [DOC(K_i, H_{i+1}) - DOC(K_i, H_i)]. \qquad (8)$$

This replication is model-independent and works as follows. If the nearest barrier H_1 is never hit, then the first option provides the necessary payoff, while the terms in the sum cancel. If H_1 is reached, then $DOC(K_0, H_1)$ and

[11] See Gastineau[10] for an introduction to rolling options.

Regions Used to Price	Price
$0 \le k \le 0$	0.80687
$-1 \le k \le 1$	0.62712
$-2 \le k \le 2$	0.62718
$-3 \le k \le 3$	0.62718
$-4 \le k \le 4$	0.62718
$-5 \le k \le 5$	0.62718

3 Month Option ($T = .25$)

Regions Used to Price	Price
$0 \le k \le 0$	0.47052
$-1 \le k \le 1$	0.03541
$-2 \le k \le 2$	0.07713
$-3 \le k \le 3$	0.07635
$-4 \le k \le 4$	0.07636
$-5 \le k \le 5$	0.07636

1 Year Option ($T = 1$)

($S = 100$, $r = 0.05$, $d = 0.03$, $\sigma = .15$, $U = 110$, $D = 90$)

Table 1: Price Convergence of Double Knockout Binary Pricing Formula.

$DOC(K_1, H_1)$ become worthless. We can re-write the remaining portfolio as:

$$RDC = DOC(K_1, H_2) + \sum_{i=2}^{n-1} [DOC(K_i, H_{i+1}) - DOC(K_i, H_i)].$$

This position is analogous to the initial one, but with initial strike K_1, and with barriers $H_2, \ldots, H_n$. If all the barriers are hit, then all the options knock out as required.

It is straightforward to replicate the payoffs of a rolldown call with vanilla options. For each down-and-out call, use (4) to find the adjusted payoff. By summing the adjusted payoffs, we can ascertain our total static hedge. Every time a barrier is reached, we need to repeat the procedure to find our new hedge portfolio. Thus, the maximum number of rebalancings is the number of barriers.

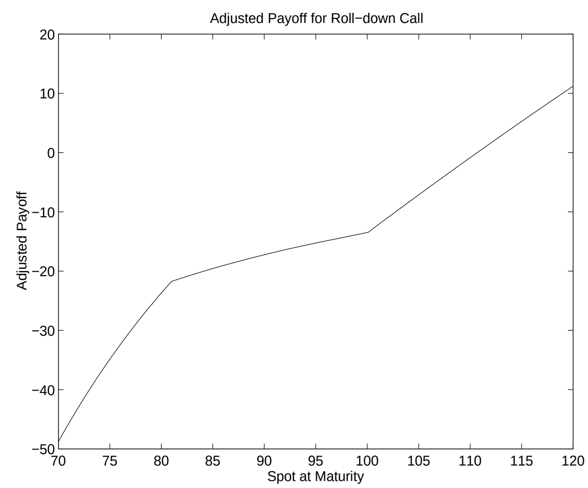
As an example, consider a rolldown call with initial strike $K_0 = 100$. Suppose it has two rolldown barriers at 90 and 80 (ie. $H_1 = 90$, $H_2 = 80$). Furhter suppose that upon hitting the 90 barrier, the strike is rolled down to the barrier (ie. $K_1 = 90$). If the spot hits 80, the option knocks out. Then, our replicating portfolio is:

$$DOC(100, 90) - DOC(90, 80) + DOC(90, 90).$$

Each of these options can be statically replicated. The sum of the corresponding adjusted payoffs is (see Figure 7):

$$f(S_T) = (S_T - 100)^+ - \left(\frac{S_T}{90}\right)^p \left(\frac{90^2}{S_T} - 100\right)^+ + \left(\frac{S_T}{80}\right)^p \left(\frac{80^2}{S_T} - 90\right)^+ - \left(\frac{S_T}{90}\right)^p \left(\frac{90^2}{S_T} - 90\right)^+$$

The portfolio will need to be rebalanced upon hitting the barriers at 90 and 80.



$(r = 0.05,\ d = 0.03,\ \sigma = .15)$

Figure 7: Adjusted payoff for Roll-down Call.

7 Ratchet Options

Ratchet calls differ from roll-down calls in only two ways. First, the strikes K_i are equal to the barriers H_i for $i = 1, \ldots, n-1$. Second, rather than knocking out at the last barrier H_n, the option is kept alive and the strike is rolled down for the last time to $K_n = H_n$. As in [7], this feature can be dealt with by replacing the last spread of down-and-out calls $[DOC(H_{n-1}, H_n) - DOC(H_{n-1}, H_{n-1})]$ in (8) with a down-and-in call $DIC(H_n, H_n)$:

$$RC = DOC(K_0, H_1) + \sum_{i=1}^{n-2} [DOC(H_i, H_{i+1}) - DOC(H_i, H_i)] + DIC(H_n, H_n).$$

Substituting in the model-free results $DOC(K, H) = C(K) - DIC(K, H)$ and $DIC(H, H) = P(H)$ simplifies the result to:

$$RC = DOC(K_0, H_1) + \sum_{i=1}^{n-2} [P(H_i) - DIC(H_i, H_{i+1})] + P(H_n). \tag{9}$$

The hedge proceeds as follows. If the forward never reaches H_1, then the $DOC(K_0, H_1)$ provides the desired payoff $(S_T - K_0)^+$ at expiration, while the puts and down-and-in calls all expire worthless. If the barrier H_1 is hit, then the $DOC(K_0, H_1)$ vanishes. The summand when $i = 1$ has the same value as a $DOC(H_1, H_2)$ and so these options should be liquidated with the proceeds used to buy this knockout. Thus the position after rebalancing at H_1 may be rewritten as:

$$RC = DOC(H_1, H_2) + \sum_{i=2}^{n-2} [P(H_i) - DIC(H_i, H_{i+1})] + P(H_n).$$

This is again analogous to our initial position. As was the case with rolldowns, the barrier options in (9) can be replaced by static positions in vanilla options. Thus, the replicating strategy for a ratchet call involves trading in vanilla options each time a lower barrier is reached.

8 Lookbacks

A lookback call is an option whose strike price is the minimum price achieved by the underlying asset over the option's life. This option is the limit of a ratchet call as all possible barriers below the initial spot are included.

A series of papers have developed hedging strategies for lookbacks which involve trading in vanilla options each time the underlying reaches a new extreme. Goldman, Sosin, and Gatto[11] were the first to take this approach. They worked within the framework of the Black Scholes model assuming $r - d = \frac{\sigma^2}{2}$. Bowie and Carr[4] and Carr, Ellis, and Gupta[7] also use a lognormal model but assume $r = d$ instead. Hobson[14] finds model-free lower and upper bounds on lookbacks. This section obtains exact replication strategies in a lognormal model with constant but otherwise arbitrary risk-neutral drift.

Our approach is to demonstrate that lookback calls or more generally lookback claims can be decomposed into a portfolio of one-touch binaries[12]. For each binary, we can create the appropriate adjusted payoffs. Thus, we can create the adjusted payoff of a lookback by combining the adjusted payoffs of the binaries. This combined adjusted payoff will give us pricing and hedging strategies for the lookback.

For simplicity, consider a lookback claim that pays off $\min(S)$. Let m be the current minimum price. At maturity, the claim will pay off:

$$m - \int_0^m bin(K)dK, \tag{10}$$

where $bin(K)$ is the payoff of a one-touch down binary struck at K. Thus, our replicating portfolio is a zero coupon bond with face value m and the continuum of dK one-touch binaries struck at $K < m$.

We can calculate the adjusted payoff of the lookback by adding the adjusted payoffs of the bond and binaries. The adjusted payoff of the bond is its face

[12] A one-touch binary pays one dollar at maturity so long as a barrier is touched at least once.

value, and the adjusted payoff of a one-touch binary with barrier K is (from (3)):

$$\hat{f}_{bin(K)}(S_T) = \begin{cases} 0 & \text{if } S_T > K, \\ 1 + (S_T/K)^p & \text{if } S_T < K, \end{cases}$$

where recall $p \equiv 1 - \frac{2(r-d)}{\sigma^2}$. Consequently, the adjusted payoff of a lookback option is:

$$\hat{f}_{lb}(S_T) = m - \int_0^m f_{bin(K)}(S_T)dK, \tag{11}$$

where $\hat{f}_{lb}(\cdot)$ and $\hat{f}_{bin(K)}(\cdot)$ are the adjusted payoffs for the lookback claim and the binary respectively.

Note that the adjusted payoff of a binary struck at K is zero for values above K. Therefore:

$$\int_0^m \hat{f}_{bin(K)}(S_T)dK = \begin{cases} \int_{S_T}^m \left[1 + (\frac{S_T}{K})^p\right] dK & \text{for } S_T < m \\ 0 & \text{for } S_T > m. \end{cases} \tag{12}$$

The integral term depends upon the value of p. In particular:

$$\int_{S_T}^m \left[1 + \left(\frac{S_T}{K}\right)^p\right] dK = \begin{cases} m - S_T + S_T \ln(m/S_T) & \text{for } p = 1 \\ m - S_T + \frac{\sigma^2}{2c} S_T((m/S_T)^{2c/\sigma^2} - 1) & \text{for } p \neq 1, \end{cases} \tag{13}$$

where $c = r - d$.

Assuming $p \neq 1$, the combination of (11), (12), and (13) implies that the adjusted payoff for the lookback claim is (see Figure 8):

$$\hat{f}_{lb}(S_T) = \begin{cases} S_T - \frac{\sigma^2}{2c} S_T((m/S_T)^{2c/\sigma^2} - 1) & \text{for } S_T < m \\ m & \text{for } S_T > m. \end{cases} \tag{14}$$

When $p = 0$ (ie. $2c = \sigma^2$), the above payoff simplifies to:

$$\hat{f}_{lb}(S_T) = \begin{cases} 2S_T - m & \text{for } S_T < m \\ m & \text{for } S_T > m. \end{cases} \tag{15}$$

In this case, the adjusted payoff is linear. Note that in all cases, the adjusted payoff is a function of m.

8.1 Hedging

As shown in (10), replicating the lookback claim involves a continuum of one-touch binaries. The hedging strategy for each binary involves rebalancing at the barrier. Thus, hedging the lookback claim involves rebalancing every time the minimum changes whcih occurs an infinite number of times. While this strategy cannot be called static, rebalancing is certainly less frequent than in the usual continuous trading strategy. In fact, the set of points where the

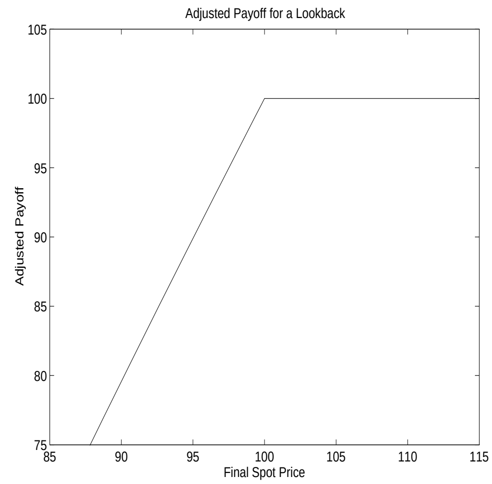


Figure 8: Adjusted payoffs for Lookback ($r = 0.05$, $d = 0.03$, $\sigma = .15$, $m = 100$).

minimum changes is almost certainly a set of measure zero[13]. We also note that our rebalancing strategy only involves at-the-money options which have high liquidity.

8.2 Lookback Variants

Lookbacks comes in many variants, and our techniques are applicable to many of them. In this subsection, we give several variants and show how they may be hedged. Let m_T denote the minimum realized spot at expiry, and let S_T denote the spot price at expiry.

- **Lookback call.** The final payoff is $S_T - m_T$. The replication involves buying the underlying and shorting the lookback claim paying the minimum at maturity. If $p = 0$, then the adjusted payoff is that of a straddle struck at m:

$$\hat{f}_{lb}(S_T) = \begin{cases} m - S_T & \text{for } S_T < m \\ S_t - m & \text{for } S_T > m. \end{cases} \tag{16}$$

  As discussed in Goldman, Sosin, and Gatto[11], the lookback value is always given by the value of a straddle struck at the minimum to date. As new minima are achieved, the strike of the straddle is rolled down via self-financing trades.

[13] In Harrison[12], it is shown that the set of times where the running minimum of a Brownian motion changes value is (almost surely) an uncountable set of measure zero.

- **Put on the Minimum.** The final payoff is $\max(K - m_T, 0)$. Let m denote the current achieved minimum. The replicating portfolio is:

$$\max(K - m, 0) + \int_0^{\min(m,K)} bin(S)dS.$$

The adjusted payoff is:

$$\hat{f}_{put-on-min} = \begin{cases} f_{lb}(\text{with } m = K) & \text{if } m > K, \\ K - m + \hat{f}_{lb} & \text{if } m < K, \end{cases}$$

where $\hat{f}_{lb}$ is the adjusted payoff of the lookback claim from (14). In the first case, we substitute $m = K$ in the formula for the adjusted payoff. Note that the adjusted payoff is fixed for $m > K$. Our hedge is static until the minimum goes below K, after which we need to rebalance at each new minimum.

- **Forward Starting Lookbacks.** These lookbacks pay m_{12}, the minimum realized price in the window from time T_1 to the maturity date T_2. In this situation, we can combine the methods from forward-starting options and lookbacks. At time T_1, we can value the lookback option with maturity T_2 as $LB(S_1)$. At initiation, we purchase a portfolio of European options with payoff $LB(S_1)$ at time T_1. At time T_1, we use the proceeds of the payoff to hedge the lookback as previously described. If $LB(S_1) = S_1 \times n(\cdot)$ where $n(\cdot)$ is independent of S_1, then the initial hedge reduces to the purchase of $n(\cdot)e^{-dT_1}$ shares. Once again, dividends are re-invested to time T_1 at which point the shares are sold and the lookback is hedged as before.

A similar analysis can be applied to the lookbacks that involve the maximum. We leave it to the reader to solve the analogous problem.

9 Summary

This paper has shown that the payoffs of several complex barrier options can be replicated using a portfolio of vanilla options which need only be rebalanced occasionally. The possible rebalancing times consist of times at which the barrier appears or disappears and at first hitting times to one or more barriers. Although all of the complex options considered can also be valued by standard techniques, the hedging strategies considered are likely to be more robust upon relaxing the standard assumptions of continuously open markets, constant volatility, and zero transactions costs.

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Appendix 1: Risk-neutral Valuation

In the Black-Scholes model, any European claim with maturity T and payoff $f(S_T)$ can be priced by taking the discounted conditional expected value of the payoff under the risk-neutral measure[15]:

$$\begin{aligned} V(S) &= e^{-rT} E^*[f(S_T)|S] \\ &= e^{-rT} \int_0^\infty f(S_T) p(S_T, S) dS_T, \end{aligned}$$

where S is the current price and

$$p(S_T, S) \equiv \frac{1}{S_T\sqrt{2\pi\sigma^2 T}} \exp\left[-\frac{(\ln(S_T/S_0) - (r - d - \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}\right]$$

is the lognormal density function. We denote the interest rate, the dividend rate, and the volatility rate by r, d, and σ, respectively.

Appendix 2: Proof of Lemma 1

Lemma 1 *In a Black-Scholes economy, suppose X is a portfolio of European options expiring at time T with payoff:*

$$X(S_T) = \begin{cases} f(S_T) & \text{if } S_T \in (A, B), \\ 0 & \text{otherwise.} \end{cases}$$

For $H > 0$, let Y be a portfolio of European options with maturity T and payoff:

$$Y(S_T) = \begin{cases} \left(\frac{S_T}{H}\right)^p f(H^2/S_T) & \text{if } S_T \in (H^2/B, H^2 A), \\ 0 & \text{otherwise} \end{cases}$$

where the power $p \equiv 1 - \frac{2(r-d)}{\sigma^2}$ and r, d, σ are the interest rate, the dividend rate, and the volatility rate respectively.

Then, X and Y have the same value whenever the spot equals H.

Proof. For any $t < T$, let $\tau = T - t$. By risk-neutral pricing, the value of X when the spot is H at time t is:

$$\begin{aligned} V_X(H, t) &= e^{-r\tau} \int_A^B f(S_T) p(H, S_T, \tau) dS_T \\ &= e^{-r\tau} \int_A^B f(S_T) \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(\ln(S_T/H) - (r - d - \frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right] dS_T. \end{aligned}$$

Let $\hat{S} = \frac{H^2}{S_T}$. Then, $dS_T = -\frac{H^2}{\hat{S}^2} d\hat{S}$ and:

$$V_X(H, t) = -e^{-r\tau} \int_{H^2/A}^{H^2/B} f(H^2/\hat{S}) \frac{1}{\hat{S}\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(\ln(H/\hat{S}) - (r - d - \frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right] d\hat{S}$$

$$= \quad e^{-r\tau} \int_{H^2/B}^{H^2/A} \left(\frac{\hat{S}}{H}\right)^p f(H^2/\hat{S}) \frac{1}{\hat{S}\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(\ln(\hat{S}/H) - (r - d - \frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right] d\hat{S},$$

where $p \equiv 1 - \frac{2(r-d)}{\sigma^2}$. By inspection, $V_X(H,t)$ exactly matches the risk-neutral valuation of Y, namely $V_Y(H,t)$. ∎

Appendix 3: Pricing Formula for Partial Barrier Call

The value of a partial barrier call with spot price S, strike K, maturity T_2, partial barrier H, and time of barrier disappearance T_1 can be computed by taking the discounted expected value under the risk neutral measure of (7) as:

$$e^{-dT_2} S M(a_1, b_2, \rho) - e^{-rT_2} K M(a_2, b_2, \rho)$$
$$-\left(\frac{S}{H}\right)^p \left[e^{-dT_2}(H^2/S) M(c_1, d_1, \rho) - e^{-rT_2} K M(c_2, d_2, \rho)\right]$$

where $p = 1 - \frac{2(r-d)}{\sigma^2}$, $M(a,b,\rho)$ denotes the cumulative bivariate normal with correlation $\rho = \sqrt{T_1/T_2}$, and

$$a_1 = \frac{\ln(S/H) + (r - d + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \qquad a_2 = a_1 - \sigma\sqrt{T_1},$$
$$b_1 = \frac{\ln(S/K) + (r - d + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, \qquad b_2 = b_1 - \sigma\sqrt{T_2},$$
$$c_1 = \frac{\ln(H/S) + (r - d + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \qquad c_2 = c_1 - \sigma\sqrt{T_1},$$
$$d_1 = \frac{\ln(H^2/SK) - (r - d + \sigma^2/2)(T_2 - T_1) + (r - d + \sigma^2/2)T_1}{\sigma\sqrt{T_2}}$$
$$d_2 = \frac{\ln(H^2/SK) - (r - d - \sigma^2/2)(T_2 - T_1) + (r - d - \sigma^2/2)T_1}{\sigma\sqrt{T_2}}.$$