Closed form Valuation of American Barrier Options*

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Abstract

Closed form formulae for European barrier options are well known from the literature. This is not the case for American barrier options, for which no closed form formulae have been published. One has therefore had to resort to numerical methods. Using lattice models like a binomial or a trinomial tree for valuation of barrier options is known to converge extremely slowly, compared to plain vanilla options. Methods for improving the algorithms have been described by several authors. However, these are still numerical methods that are quite computer intensive. In this paper we show how American barrier options can be valued analytically in a very simple way. This speeds up the valuation dramatically as well as give new insight into barrier option valuation.

1 Analytical valuation of American barrier options

Closed form solutions and valuation techniques for standard European barrier options are well known from the literature, see for instance (Merton 1973, Reiner and Rubinstein 1991, Rich 1994, Haug 1997). No closed form solution for American barrier options exists in the extant literature. The technique used to value American barrier options have therefore been numerical methods. Lattice models have been especially popular. Without doing any adjustments lattice models have been shown to convergence extremely slowly. Several methods have been published to improve on the technique (Boyle and Lau 1994, Ritchken 1994, Derman, Bardhan, Ergener, and Kani 1995), but the method is still quite computer intensive. In the present paper we suggest an analytical solution. This offers both to speed up the valuation process and it gives new insight into the valuation of barrier options.

We limit ourselves to assume that the underlying asset follows a geometric Brownian motion without drift in the risk adjusted economy (i.e., we consider the process after an appropriate change of probability measure). Futures and forwards contracts are examples of underlying securities that satisfy this restriction.

$$dS_t = \sigma S_t dz_t$$

S is the asset price, σ is the instantaneous standard deviation of the rate of return, and dz is a standard Wiener process.

The idea is to use the reflection principle described by e.g. Harrison (1985). In a barrier context (e.g. a down-and-in call) the reflection principle basically states that the number of paths leading from S_t to a point higher than X that touch a barrier level H ($H < S_t$) before maturity is equal to the number of paths from an asset that starts from H^2/S_t. Using the reflection principle we can then simply value both European and American barrier options on the basis of formulas from plain vanilla options.

Using the reflection principle the value of a European or American down-and-in call is equal to (assuming $H < S_t$):

$$C_t^{di}(S_t, X, H, T, r, b, \sigma) = \frac{S_t}{H} C_t\left(\frac{H^2}{S_t}, X, T, r, b, \sigma\right) = C_t\left(H, \frac{S_t X}{H}, T, r, b, \sigma\right),$$

where $C_t^{di}(S_t, X, H, T, r, b, \sigma)$ is a call down-and-in (the superscript indicating the type of barrier option di =down-and-in) with asset price S_t, strike X, barrier H, time to maturity in years T, risk free rate r, cost of carry $b = 0$, and volatility σ. $C_t\left(H, \frac{S_t X}{H}, T, r, b, \sigma\right)$ is a plain vanilla American call with asset price equal to H and strike price equal to $\frac{S_t X}{H}$. For European barrier options we could naturally just replace the American plain vanilla call with a European c_t. This implies that all we need to value an American down-and-in call analytically is a closed form solution for a plain vanilla American call option. This involves using a closed form approximation, like for instance the popular closed form model of Barone-Adesi and Whaley (1987), or the closed form method of Bjerksund and Stensland (1993). Similarly, using the reflection principle, the value of a European or American up-and-in put can be shown to be equal to

$$P_t^{ui}(S_t, X, H, T, r, b, \sigma) = P_t\left(H, \frac{S_t X}{H}, T, r, b, \sigma\right)$$

If we know how to value knock-in options then the value of a knock-out option can easily be found by using the well known out-in barrier parity:

$$\text{Out-option} = \text{long plain vanilla option} + \text{short in-barrier option}$$

In other words, we have all we need to value most types of standard American barrier options analytically.

2 Numerical comparison

In this section we will compare some well known methods for barrier option valuation with our closed form solution method.

Table 1 compares European barrier option values. Column one is calculated using the closed form barrier formulas derived by Reiner and Rubinstein (1991). Column two is calculated using the formula of Black (1976) in combination with the reflection principle. As expected, these two columns contain identical values. Column three and four contain values calculated using a trinomial tree without any adjustments. It is evident that using a tree without any corrections is more or less useless. Column five and six are calculated using the trinomial tree of Boyle (1986) in combination with the barrier technique developed by Derman, Bardhan, Ergener, and Kani (1995). Using 300 time steps this method gives quite accurate values, except when the barrier is very close to the asset price. The last column is based on the binomial tree of Cox, Ross, and Rubinstein (1979) in combination with the barrier technique described by Boyle and Lau (1994). The Boyle-Lau method does not allow direct control of the number of time steps. The method instead offers choices of the optimal number of time steps. The numbers in brackets are the number of time steps used. We have chosen to have the number of time steps equal to the first number higher than 100 of the time steps given by the Boyle-Lau formula . As can be seen from the table the Boyle-Lau method gives accurate values in all cases. However, the number of time steps have to be extremely large (1421) when the barrier is very close to the asset price ($H = 94$).

American barrier option values are compared in table 2. The first column is calculated using the closed form approximation method suggested by Barone-Adesi and Whaley (1987) in combination with the reflection principle.

Table 1: Comparison of European down-and-out call barrier option values
($S_t = 94.5$, $X = 105$, $T = 1$, $r = 0.10$, $b = 0$, $\sigma = 0.20$)

H	Barrier formula	Black-76 reflection	Plain tree 50 steps	Plain tree 300 steps	Derman 50 steps	Derman 300 steps	Boyle-Lau binomial
94	0.2769	0.2769	1.6972	0.7986	0.8488	0.3850	0.2770 (1421)
93	0.7837	0.7837	1.6972	0.7986	0.9673	0.7664	0.7854(156)
90	1.9543	1.9543	2.6474	1.9610	1.9704	1.9407	1.9585(151)
85	2.9788	2.9788	3.1149	3.0589	2.8649	2.9299	2.9753(128)

Table 2: Comparison of American down-and-out call barrier option values
($S_t = 94.5$, $X = 105$, $T = 1$, $r = 0.10$, $b = 0$, $\sigma = 0.20$)

H	BAW reflection	Plain tree 50 steps	Plain tree 300 steps	Derman 50 steps	Derman 300 steps	Boyle-Lau binomial
94	0.2860	1.7348	0.8181	0.8702	0.3950	0.2841(1421)
93	0.8091	1.7348	0.8181	0.9912	0.7852	0.8046(156)
90	2.0166	2.7001	2.0038	2.0133	1.9831	2.0015(151)
85	3.0752	3.1730	3.1172	2.9204	2.9868	3.0342(128)

Also in this case is the unadjusted trinomial tree more or less useless, as it is extremely slow to converge. Both the method of Boyle and Lau (1994) and the method of Derman, Bardhan, Ergener, and Kani (1995) work fine as long as the barrier is not too close to the asset price. The reflection principle is an analytical solution. It is therefore naturally much faster then the lattice models. The closed form reflection principle should thus be of great interest to valuate American barrier options on futures and forwards, when assuming geometric Brownian motion. The accuracy of the model will naturally depend on the accuracy of the plain vanilla American option formula used.

It is however worth noting that our approach will not work in general, when one moves away from the assumption of geometric Brownian motion, or when working with complex barrier options. For instance, when working with an implied tree model calibrated to the volatility smile found in the market, the only available methodology is still numerical methods (see e.g. (Dupire 1994, Derman and Kani 1994, Rubinstein 1994)). The method of Boyle and Lau (1994) will in general only work on a standard Cox-Ross-Rubinstein (CRR) tree. On the other hand, the method of Derman, Bardhan, Ergener, and Kani (1995) is very flexible and independent of the underlying tree model (binomial, trinomial, multinomial, implied trees). This makes theirs the method of choice when valuing complex barrier options.

The method of Boyle and Lau (1994) is basically built for barrier valuation in a CRR binomial tree. This implies an additional weakens of their method. In situations when the risk-free rate is very high and the volatility is very low the CRR tree can actually give negative probabilities. In most practical situations this is not a problem, but it could certainly happen in special market situations.

3 Conclusion

We have shown how to price American barrier options using a plain vanilla American option formula, utilizing the reflection principle. This enables fast and accurate valuation of American barrier options. For valuation of more complex barrier options numerical solutions are still the only game in town. Directions for further research in this field is for this reason naturally to try to extend our results to also hold for valuation of more complex forms of barrier options.

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