PRICING OF ASIAN EXCHANGE RATE OPTIONS UNDER STOCHASTIC INTEREST RATES AS A SUM OF DELAYED PAYMENT OPTIONS

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Abstract. The aim of the paper is to develop pricing formulas for European type Asian options written on the exchange rate in a two currency economy. The exchange rate as well as the foreign and domestic zero coupon bond prices are assumed to follow geometric Brownian motions. As a special case of a discrete Asian option we analyse the delayed payment currency option and develop closed form pricing and hedging formulas.

The main emphasis is devoted to the discretely sampled Asian option. It is shown how the value of this option can be approximated as the sum of Black-Scholes options. The formula is obtained under the application of results developed by Rogers and Shi (1995) and Jamshidian (1991). In addition bounds for the pricing error are determined.

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1. Introduction

No Asian option is traded as a standardized contract in any organized exchange. However, they are extremely popular in the OTC market among institutional investors. Milevsky and Posner (1998) mention that the estimated outstanding volume is in the range from five to ten billion USD.

Several reasons for introducing Asian options are presented in the literature. A corporation expecting to have payments in foreign currency claims can reduce its average foreign currency exposure by using Asian options. Because the average, which is the underlying asset in these contracts, tends to be less volatile than the exchange rate itself, the Asian option is (normally) priced more cheaply than the standard option. The hedging costs for the firm is therefore reduced.

Another reason for introducing Asian options was to avoid speculators in arranging price manipulation of the underlying asset close to the maturity date.

The value of an Asian option depends at any point in time on the spot exchange rate and on the history of the spot exchange rate: the Asian option is path–dependent. This increases the complexity of both pricing and hedging. In addition the probability distribution of the arithmetic average is unknown if we, as is usually the case, assume that the exchange rate and the relevant bond prices follow standard lognormal processes. Numerical techniques must be relied on in order to determine the prices of Asian options in general, and this brings us in a difficult position to generate the hedging strategy. The hedging strategy is determined through the price sensitivity of the Asian option to changes in the average. However, as the average itself is not a traded asset we need to develop how the average can be duplicated through a self financing strategy in traded assets. The main problem is clearly that we do not have a closed form solution giving us the price of the Asian option, we have at best a good approximation. No guarantee exists, however, that the sensitivity of the approximation is just similar or close to the sensitivity of the true price.

The numerical techniques applied for valuing Asian options are numerous. A majority of the methods can be extended to the situation we consider with a stochastic developing term structure in both the domestic and the foreign country. With the symbol * Table 1 indicates those methods which in their conception are applicable to the valuation problem under stochastic interest rates*. Concerning the

*See e.g. Nielsen and Sandmann (1996) and Schmidli (1997).

Table 1. Valuation methods for Asian Options

	Method	References
*	Monte Carlo Simulation	Kemna and Vorst (1990)
*	Approximation via geometric average	Vorst (1992)
*	Approximate the probability density function of the arithmetic average by a reciprocal gamma distribution	Milevsky and Posner (1998)
	Laplace transformation	Geman and Yor (1993)
*	Edgeworth expansion	Turnbull and Wakeman (1991) Levy (1992) Jacques (1996)
	Binomial trees	Hull and White (1993)
	Fast Fourier Transformation	Carverhill and Clewlow (1990)
*	Numerical solution of the variable reduced second order p.d.e.	Rogers and Shi (1995) Alziary, Décamps and Koehl (1997) He and Takahashi (1996)

hedging of Asian options only a few papers exist. In Alziary, Décamps and Koehl (1997) hedging strategies are analysed in the situation where the term structure of interest rates develops in a deterministic manner, and this can obviously not be the relevant case for exchange rate options. Alziary et al use a variable reduction approach introduced by Rogers and Shi (1995) to price the Asian option and they continue showing that the option's delta can be derived from this pde. In their paper they furthermore derive useful expressions for the difference between the European and the Asian option. Turnbull and Wakeman (1991) give a rough estimate of the delta based on an Edgeworth series expansion. El Karoui and Jeanblanc-Picquè (1993) show that the price of an Asian option on a stock is equal to a European option on a fictitious asset which has a random volatility. They propose a super hedging strategy based on Black and Scholes (1973) formula with volatility equal to the upper bound on the stochastic volatility. Jacques (1996) uses approximate price formulas based on the lognormal approximation and based on an inverse Gaussian approximation. In both situations he derives a formula for the hedging portfolio and shows through numerical examples that these formulae are efficient in the sense that the replicating strategy is close to the intrinsic value of the Asian option at maturity. Jacques finds, comparing the two approximations, that they are equally efficient.

From the mathematical point of view the main difficulty of pricing and hedging an Asian option is to determine the distribution of the arithmetic average. Some of the mentioned approximation techniques can be interpreted as a change to a more convenient distribution. On one hand this seems to be a reasonable approach for two reasons. First, the continuously growing literature in finance based on the lognormality assumption is in no way a verification of this distribution for the changes of a financial asset. Second, empirical evidence for this distribution is certainly doubtful, but a uniformly better distribution cannot be identified. On the other hand the objective of financial modeling does amount to something more than the calculation of numbers. The objective is to clarify dependencies. To precisely measure the size of these dependencies, we have to study examples, i.e. to specify distributions.

A theoretical model should be understood as a reference model. In financial markets the relevance of a reference model for practical purposes is whether or not it is accepted and serves as a guideline to clarify relationships and the impact of decisions. The Black and Scholes (1973) model has become the most widely accepted model for the analysis of derivative assets. Although theorists and practitioners are aware of its weaknesses, this model is used as the reference model in finance. With a view to the mathematical problems in the situation of Asian options the tentative to leave the Black and Scholes model may be strong, but the consistency with results already understood and recognized seems more important to us.

The problem of pricing and hedging an Asian option proves to be much more difficult when the bond markets are described by a model allowing for stochastic term structure developments. It is the extension to such a situation which will be the aim of this paper.

Following the description in Section 2 of the financial market model to be used in this paper, we analyse in Section 3 the pricing of Asian options and generalize the Rogers and Shi approach to the situation of an Asian exchange rate option with stochastic interest rates, i.e. we have to take into consideration the time dependent, multi–dimensional volatility structure. Applying a result from Jamshidian (1991) we derive an analytical closed form solution for the approximation.

Section 4 is devoted to numerical analysis of the closed form solution. Finally, we conclude in Section 5.

2. The financial market model

The model of the financial market we consider is a two country model. The exchange rate between the domestic and the foreign country is assumed to be stochastic. In addition we consider a stochastic behavour of the interest rate market in both countries, and apply the bond price oriented approach by Geman, El Karoui and Rochet (1995). The model is based on the international financial market model derived by Amin and Jarrow (1991), and we restrict ourselves to the situation with deterministic volatility functions. As discussed in the introduction, existing results presume deterministic interest rates. In the case of the Asian exchange rate option this assumption seems questionable to us. In order to compare these results with those implied by a stochastic behavour of the interest rate markets we are staying within the class of lognormal processes.

The continuous time model is defined on the finite time interval $[0, T]$. Let $(\Omega, \mathcal{F}, P_d^*)$ be a filtered probability space where the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by an n-dimensional ($n \geq 3$) Brownian motion $\{W_d^*(t)\}_{t\in[0,T]}$ under P_d^*. Following Geman, El Karoui and Rochet (1995), the domestic interest rate market is characterized by the price process of the family of the domestic zero coupon bonds, i.e.

Assumption 2.1:

For any maturity $t' \in [0, T]$ the price process $\{D^d(t, t')\}_{t\in[0,t']}$ of the domestic default free zero coupon bond with domestic value 1 and maturity t' satisfies the stochastic differential equation

$$
\begin{aligned}
dD^d(t, t') &= r^d(t) D^d(t, t') dt + D^d(t, t') \sigma^d(t, t') \cdot dW_d^*(t) \quad \forall\ t \in [0, t'] \qquad (2.1) \\
D^d(t', t') &= 1 \quad P_d^* \quad \textit{a.s. in the domestic currency,}
\end{aligned}
$$

where $\{r^d(t)\}_{t\in[0,T]}$ *denotes the domestic continuously compounded spot rate process*† *and the volatility structure satisfies the following requirements:*

i) *for any* $t' \in [0,T]$ *the volatility function*

$$\sigma^d(\cdot,t') : [0,t'] \to \mathbb{R}^n$$

is continuous and square integrable on $[0,t']$ *with* $\sigma(t',t') = 0$,

ii) $\left|\left|\frac{\partial\sigma^d(t,t')}{\partial t'}\right|\right|$ *is bounded on* $\{(t,t')|0 \le t \le t', t' \in [0,T]\}$,

iii) *there exists a real number* $H > 0$ *such that*

$$\left|\left|\frac{\partial\sigma^d(t,t'+\delta)}{\partial t'} - \frac{\partial\sigma^d(t,t')}{\partial t'}\right|\right| \le H\cdot\delta \quad \forall\ t \le t' \quad \forall\ \delta > 0.$$

With Assumption 2.1 the domestic interest rate market is arbitrage free and P_d^* is the unique (domestic) martingale measure (see Geman, El Karoui and Rochet (1995)). The solution for the domestic zero coupon bond is given by

$$(2.2)\qquad D^d(t,t') \;=\; D^d(0,t')\exp\left\{\int_0^t r^d(u)du - \frac{1}{2}\int_0^t ||\sigma^d(u,t')||^2 du + \int_0^t \sigma^d(u,t')\cdot dW_d^*(u)\right\} \quad .$$

As in Frey and Sommer (1996), the foreign interest rate market and the exchange rate are modeled under the domestic martingale measure P_d^*, i.e. with reference to the contribution by Amin and Jarrow (1991), we assume

Assumption 2.2:

a) *For all* $t' \in [0,T]$ *the foreign zero coupon bond market is determined by*

$$(2.3)\qquad dD^f(t,t') \;=\; \left[r^f(t) - \sigma^f(t,t')\cdot\sigma^x(t)\right]D^f(t,t')dt + D^f(t,t')\sigma^f(t,t')\cdot dW_d^*(t)$$

$$D^f(t',t') \;=\; 1 \qquad P_d^* \quad \textit{a.s. in the foreign currency,}$$

†By $x\cdot y$ for $x,y\in\mathbb{R}^n$ we denote the standard scalar product, i.e. $x\cdot y = \sum_{i=1}^n x_i y_i$ and $||x|| := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$ denotes the Euclidian norm.

b) *The exchange rate process* $\{X(t)\}_{t\in[0,T]}$ *in units of the domestic currency per one unit of the foreign currency satisfies*

$$dX(t) \quad = \quad \left[r^d(t) - r^f(t)\right] X(t)dt + X(t)\sigma^x(t)\cdot dW_d^*(t) \quad , \tag{2.4}$$

where $\{r^f(t)\}_{t\in[0,T]}$ *is the process of the foreign spot rate and the volatility functions are assumed to satisfy the same requirements as in Assumption 2.1.*

Given the spot rate processes in both countries the bank account is defined by

$$\beta_{t,T}^d \quad := \quad \exp\left\{\int_t^T r^d(u)du\right\} \quad \text{for the domestic country, resp. by} \tag{2.5}$$

$$\beta_{t,T}^f \quad := \quad \exp\left\{\int_t^T r^f(u)du\right\} \quad \text{for the foreign country.}$$

Obviously, the expected discounted value of the foreign zero coupon bond is not a martingale under P_d^*, since P_d^* is the domestic martingale measure. But for the process $\left\{Z^d(t,t') := X(t)D^f(t,t')\right\}_{t\in[0,t']}$, i.e. the value of the foreign zero coupon bond denoted in the domestic currency, this is true, since

$$dZ^d(t,t') \quad = \quad r^d(t)Z^d(t,t')dt + Z(t,t')\left[\sigma^x(t) + \sigma^f(t,t')\right]\cdot dW_d^*(t) \tag{2.6}$$

$$Z^d(t',t') \quad = \quad X(t')D^f(t',t') \qquad P_d^* \quad a.s. \text{ in the domestic currency.}$$

It could be argued that the geometric Brownian representation chosen for all the specified stochastic processes has the drawback that it allows for negative interest rates. However, the pricing approach taking in this paper will only be feasible under this choice. Furthermore, the drift and the volatility terms can be chosen in such a manner that the probability of negative interest rates, within the maximal time to maturity of an Asian exchange rate option, is low.

The stochastic differential equations for the domestic and foreign bonds and for the exchange rate may alternatively be expressed using the forward risk adjusted measure. By Girsanov's Theorem the domestic T-forward risk adjusted measure P_d^T is defined as an equivalent measure with respect to P_d^* with the Radon-Nikodym derivative

$$\left.\frac{dP_d^T}{dP_d^*}\right|_t \quad = \quad \frac{(\beta_{t,T}^d)^{-1}D^d(T,T)}{E_{P_d^*}[(\beta_{t;T}^d)^{-1}D^d(T,T)|\mathcal{F}_t]} = \exp\left\{-\frac{1}{2}\int_t^T ||\sigma^d(u,T)||^2du + \int_t^T \sigma^d(u,T)\cdot dW_d^*(u)\right\}$$

where $dW_d^T(t) = dW_d^*(t) - \sigma^d(t,T)dt$ defines a vector Brownian motion under P_d^T. With the domestic zero coupon bond $D^d(t,T)$ resp. the foreign bond $D^f(t,T)$ as a numeraire, the stochastic differential equations (2.1), (2.3) and (2.4) can be rewritten as

$$
\begin{aligned}
d\left(\frac{D^d(t,t')}{D^d(t,T)}\right) &= \frac{D^d(t,t')}{D^d(t,T)}\eta^d(t,t',T)\cdot dW_d^T(t) \\
d\left(\frac{D^f(t,t')}{D^f(t,T)}\right) &= \frac{D^f(t,t')}{D^f(t,T)}\eta^f(t,t',T)\cdot [dW_d^T(t) - \eta^x(t,T)dt] \qquad (2.7)\\
d\left(\frac{X(t)D^f(t,T)}{D^d(t,T)}\right) &= \frac{X(t)D^f(t,T)}{D^d(t,T)}\eta^x(t,T)\cdot dW_d^T(t)
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad \eta^d(t,t',T) &:= \sigma^d(t,t') - \sigma^d(t,T) \\
\eta^f(t,t',T) &:= \sigma^f(t,t') - \sigma^f(t,T) \\
\eta^x(t,T) &:= \sigma^x(t) + \sigma^f(t,T) - \sigma^d(t,T) \quad .
\end{aligned}
$$

The solutions for the domestic and the foreign zero coupon bonds under the domestic T-forward risk adjusted measure imply that the solution for the exchange rate process can be written as

$$
\begin{aligned}
(2.8) \qquad X(t) &= X(0)\frac{D^f(0,T)}{D^f(t,T)}\frac{D^d(t,T)}{D^d(0,T)}\exp\left\{-\frac{1}{2}\int_0^t ||\eta^x(u,T)||^2du + \int_0^t \eta^x(u,T)\cdot dW_d^T(u)\right\} \\
&= X(0)\frac{D^f(0,t)D^d(t,t)}{D^d(0,t)D^f(t,t)} \\
&\quad \cdot\exp\left\{-\frac{1}{2}\int_0^t \eta^x(u,t)\cdot(\eta^x(u,t) + 2\eta^d(u,t,T))du + \int_0^t \eta^x(u,t)\cdot dW_d^T(u)\right\}
\end{aligned}
$$

The continuous time arithmetic average exchange rate in the domestic currency is defined as

$$
(2.9) \qquad A_c(T) := \frac{1}{T}\int_0^T X(u)D^f(u,u)du,
$$

whereas for given averaging times $\{t_1 < \ldots < t_N = T\}$ the discrete average is determined by

$$
(2.10) \qquad A_d(T) := \frac{1}{N}\sum_{i=1}^N X(t_i)D^f(t_i,t_i) \quad .
$$

Setting $n(t) = \max\{i|t_i < t\}\forall t \geq t_1$ and $n(t) = 0$ for $t < t_1$ the value of an asset with the payment $A_d(T)$ at time T is determined by the expected discounted discrete average under the domestic martingale measure, i.e.

$$
\begin{aligned}
(2.11)\qquad \Pi_t(A_d(T)) &= E_{P_d^*}[(\beta_{t,T}^d)^{-1}A_d(T)|\mathcal{F}_t] \\
&= E_{P_d^T}[D^d(t,T)A_d(T)|\mathcal{F}_t] \\
&= \frac{D^d(t,T)}{N}\left[\sum_{i=1}^{n(t)}\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)} + \sum_{i=n(t)+1}^{N} E_{P_d^T}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\middle|\mathcal{F}_t\right]\right] \\
&= \frac{D^d(t,T)}{N}\left[\sum_{i=1}^{n(t)}\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right. \\
&\quad \left. + X(t)\sum_{i=n(t)+1}^{N}\frac{D^f(t,t_i)}{D^d(t,t_i)}\exp\left\{-\int_t^{t_i}\eta^x(u,t_i)\cdot\eta^d(u,t_i,T)du\right\}\right]
\end{aligned}
$$

The first term of (2.11) just covers the known average at time t determined by the exchange rates prior to time t. For $t=0$ equation (2.11) simplifies to

$$
(2.12)\qquad \Pi_0(A_d(T)) = \frac{X(0)D^d(0,T)}{N}\sum_{i=1}^{N}\frac{D^f(0,t_i)}{D^d(0,t_i)}\exp\left\{-\int_0^{t_i}\eta^x(u,t_i)\cdot\eta^d(u,t_i,T)du\right\},
$$

which implies for $\frac{T}{N} =: \Delta t \to 0$ that the value of the continuous time average is equal to[‡]

$$
\begin{aligned}
(2.13)\quad \lim_{\Delta t\to 0}\Pi_0(A_d(T)) &= \frac{X(0)D^d(0,T)}{T}\int_0^T\left(\frac{D^f(0,v)}{D^d(0,v)}\exp\left\{-\int_0^v\eta^x(u,v)\cdot\eta^d(u,v,T)du\right\}\right)dv \\
&= D^d(0,T)E_{P_d^T}\left[\frac{1}{T}\int_0^T\frac{X(v)D^f(v,v)}{D^d(v,v)}dv\right] = \Pi_0(A_c(T)) \quad .
\end{aligned}
$$

Observe that $\eta^d(u,t_i,T)$ equals 0 if the domestic interest rate is deterministic, which will give us a model which does not depend on the volatility structure. If, on the other hand, the domestic market is stochastic but the foreign market is deterministic, we will still have the influence from the volatility structure. This means that the domestic and the foreign bond market enters the model in a highly nonsymmetric manner. This will be analysed further in Section 4, devoted to the numerical analysis.

[‡]As in Vorst (1992) the expected continuous time average in the case of deterministic interest rates and a flat yield curve is given by

$$
\begin{aligned}
\lim_{\Delta t\to 0}\Pi_0(A_d(T)) &= X(0)\frac{e^{-r^dT}}{T}\int_0^T e^{-(r^f-r^d)v}dv \\
&= \frac{X(0)}{T}\frac{1}{r^d-r^f}\left[e^{-r^fT}-e^{-r^dT}\right]\underset{T\to\infty}{\longrightarrow} 0
\end{aligned}
$$

3. Pricing of Asian Options: The Approximation Method

The calculation of the value of an Asian option and the hedge ratio requires the application of numerical methods and the specification of the volatility functions. With the Assumptions 2.1 and 2.2, the volatility functions of the exchange rate process and the domestic and foreign zero coupon bond markets are restricted to be non-stochastic. As a consequence, the discretely or continuously sampled arithmetic average of the exchange rate represents a weighed average of multi-dimensional lognormally distributed variables. A variety of numerical approaches for the pricing of Asian options can be considered in the case of one-dimensional lognormally distributed variables with constant volatility. Obviously these methods must be generalized in severals ways, since our situation implies time-dependent volatility functions, a complicated correlation structure and a multi-dimensional distribution.

As already discussed in a similar but less complex situation by Nielsen and Sandmann (1996) the Fast Fourier transformation technique applied in a paper by Carverhill and Clewlow (1990) to the pricing of Asian options can not be generalized to the situation of stochastic interest rates. The Edgeworth expansion which was applied by Turnbull and Wakeman (1991) involves complicated calculations of the first four moments of the arithmetic average. In the case of stochastic interest rates these calculations must be done recursively and hence by a slow algorithm similar to the one proposed by Nielsen and Sandmann. The Laplace transformation approach introduced by Geman and Yor (1993) is associated with the numerical inversion of a non-trivial Laplace transformation. Although this approach is of high mathematical elegance, the generalization of this method to the multi-dimensional case does not lead to a profound economic interpretation. Finally, it seems to us that a multinomial approximation based on the binomial approach of Hull and White (1993) is not appropriate in our case. Therefore, we will not consider these methods.

The analysis performed in this paper will involve an extension of the Rogers and Shi (1995) approach.

Rogers and Shi propose an approximation in the case of non stochastic interest rates. In addition, they are able to derive upper and lower bounds for the price of an Asian option. Since the method is very efficient from the numerical point of view in the one-dimensional case, it seems to be interesting to generalize the method to our situation and to ask for some economic interpretation.

For simplicity of the exposition we restrict ourselves to the fixed strike Asian option. The floating strike case is covered if we apply a further change of measure, i.e. by choosing $X(T)D^f(T,T)$ as a numeraire. Note, that by

$$\left.\frac{dP_d^T}{dP_X}\right|_t = \frac{X(T)D^f(T,T)}{E_{P_d^T}[X(T)D^f(T,T)|\mathcal{F}_t]}$$

$$= \exp\left\{-\frac{1}{2}\int_t^T ||\eta^x(u,T)||^2\,du + \int_t^T \eta^x(u,T)\cdot dW_d^T(u)\right\} \tag{3.1}$$

the new probability measure is defined. Furthermore, $dW_X(t) = dW_d^T - \eta^x(t,T)dt$ defines a vector Brownian motion under P_X and the arbitrage price of a floating strike Asian option can be expressed by

$$E_{P_d^*}\left[\left(\beta_{t,T}^d\right)^{-1}[X(T)-A_l(T)]^+|\mathcal{F}_t\right] = X(0)D(0,T)E_{P_X}\left[[1-\tilde{A}_l(T)]^+|\mathcal{F}_t\right] \tag{3.2}$$

with $\tilde{A}_d(T) = \frac{1}{N}\sum_{i=1}^N \frac{X(t_i)}{X(T)}$ resp. $\tilde{A}_c(T) = \frac{1}{T}\int_0^T \frac{X(u)}{X(T)}du$ and

$$\frac{X(t)}{X(T)} = \frac{D^f(0,t)}{D^f(0,T)}\frac{D^d(0,T)}{D^d(0,t)}\exp\left\{-\int_0^t \eta^x(u,t)\cdot\eta^d(u,t,T)du\right\}$$

$$\cdot\exp\left\{-\frac{1}{2}\int_0^T ||\eta^x(u,t)1_{u\le t} - \eta^x(u,T)||^2\,du + \int_0^T (\eta^x(u,t)1_{u\le t} - \eta^x(u,T))\cdot dW_X(u)\right\} \quad . \tag{3.3}$$

Thus the floating strike case can be solved as the fixed strike case by changing from the domestic T-forward risk adjusted measure to the measure P_X and adjusting the volatility functions.

3.1. **The Conditional Expectation Approach.** In their paper on the value of an Asian option, Rogers and Shi (1995) derive an approximation by use of a conditional expectation. The computation of the approximation can be done very fast numerically. In addition, the approximation error turns out to be quite small for an appropriate choice of the conditioning variable. This very nice approximation was introduced by Rogers and Shi for the Black and Scholes (1973) framework.

The objective of this section is to generalize the Rogers and Shi approach to the situation of an Asian exchange rate option with stochastic interest rates, i.e. we have to take care of the time dependent, multi-dimensional volatility structure. The generalization proves to be straightforward. Beyond the original Rogers and Shi approximation we derive an analytical closed form solution for the approximation. This closed form solution can be interpreted as a portfolio of European type options. Of course the results can be applied to the Black-Scholes model, i.e. to the case of deterministic interest rates and constant volatility for the underlying asset.

Whether the approximation is useful or not depends on the size of the approximation error. The latter depends on the choice of the conditioning random variable. The choice of this random variable is of course related to the specific contract under consideration. Like Rogers and Shi (1995) we can not determine the conditioning variable which minimizes the approximation error. Instead, we follow their argument and motivate a specific choice.

Let Z be a random variable, $A_l(T)$ the discretely or continuously sampled arithmetic average and K the fixed strike of the Asian option. The forward value of an Asian option is equal to the expected value of the terminal payoff under the domestic T-forward risk adjusted measure P_d^T. For simplicity of the expressions, denote by $E^t[.]$ the conditional expectation with respect to the σ-algebra $\mathcal{F}_t$. As in Rogers and Shi (1995) this expectation satisfies the following relation

$$E^t_{P_d^T}\left[[A_l(T)-K]^+\right] = E^t_{P_d^T}\left[E^t_{P_d^T}\left[[A_l(T)-K]^+|Z\right]\right] \geq E^t_{P_d^T}\left[\left[E^t_{P_d^T}[A_l(T)-K|Z]\right]^+\right]. \tag{3.4}$$

For the fixed strike Asian option[§] the difference between the unconditional expectation and the lower bound, i.e. the forward value of the approximation error can be estimated by ε_{RS} :

$$\begin{aligned}
0 &\leq E^t_{P_d^T}\left[E^t_{P_d^T}\left[[A_l(T)-K]^+|Z\right] - \left[E^t_{P_d^T}[A_l(T)-K|Z]\right]^+\right] \\
&= \frac{1}{2}E^t_{P_d^T}\left[E^t_{P_d^T}\left[|A_l(T)-K|\ \ |Z\right] - \left|E^t_{P_d^T}[A_l(T)-K|Z]\right|\right] \\
&\leq \frac{1}{2}E^t_{P_d^T}\left[E^t_{P_d^T}\left[\ \left(\left|A_l(T)-K-E^t_{P_d^T}\left[A_l(T)-K|Z\right]\right|\right)\ \middle|Z\right]\ \right] \\
&\leq \frac{1}{2}E^t_{P_d^T}\left[\left(V_{P_d^T}[A_l(T)|Z]\right)^{\frac{1}{2}}\right] \leq \frac{1}{2}\left(E^t_{P_d^T}\left[V_{P_d^T}[A_l(T)|Z]\right]\right)^{\frac{1}{2}} =: \varepsilon_{RS}
\end{aligned} \tag{3.5}$$

Proposition 3.1:

Let Z be a one-dimensional standard normally distributed random variable and suppose that the exchange rate and the domestic and foreign zero coupon bond markets satisfy the Assumption 2.1 and 2.2 resp. Denote by $K(t)$ the difference between the strike K and the known average at t, i.e. $K(t) := K - \frac{t}{T}A_c(t)$ or $K(t) := K - \frac{n(t)}{N}A_d(t)$ resp.. The lower bound for an Asian exchange rate option with $K(t) > 0$ is

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[§]The same argument can be applied to the floating strike Asian option. Instead of the measure P_d^T the problem should be considered under the measure P_X (see (3.1)).

given by

$$D^d(t,T)E^t_{P_d^T}[[A_d(T)-K]^+] \geq D^d(t,T)E^t_{P_d^T}\left[\left[\left(\frac{1}{N}\sum_{i=n(t)+1}^{N}F(t,t_i)\exp\left\{m_t(t_i)Z+\frac{1}{2}\nu_t^2(t_i,t_i)\right\}\right)-K(t)\right]^+\right] \tag{3.6}$$

in the case of a discrete Asian option, and

$$D^d(t,T)E^t_{P_d^T}[[A_c(T)-K]^+] \geq D^d(t,T)E^t_{P_d^T}\left[\left[\frac{1}{T}\int_t^T F(t,u)\exp\left\{m_t(u)Z+\frac{1}{2}\nu_t^2(u,u)\right\}du-K(t)\right]^+\right] \tag{3.7}$$

in the case of a continuous Asian option, where

$$F(t,\tau) := X(t)\frac{D^f(t,\tau)}{D^d(t,\tau)}\exp\left\{-\frac{1}{2}\int_t^\tau \eta^x(u,\tau)\cdot\left[\eta^x(u,\tau)+2\eta^d(u,\tau,T)\right]du\right\} \in \mathbb{R}$$

$$\nu_t^2(\tau,s) := \int_t^{\min\{s,\tau\}} \eta^x(u,\tau)\cdot\eta^x(u,s)du - m_t(\tau)m_t(s) \in \mathbb{R}$$

$$m_t(\tau) := E^t_{P_d^T}\left[Z\int_t^\tau \eta^x(u,\tau)\cdot dW_d^T(u)\right]$$

Proof. For simplicity of the proof we consider the situation $t=0$ and omit the subscript t. The assumptions on Z imply for the conditional expectation

$$E_{P_d^T}\left[\int_0^t \eta^x(u,t)\cdot dW_d^T(u)\middle| Z\right] = E_{P_d^T}\left[Z\int_0^t \eta^x(u,t)\cdot dW_d^T(u)\right]Z = m(t)Z$$

and for the conditional covariance

$$\begin{aligned}\nu^2(s,t) &:= \mathrm{cov}_{P_d^T}\left(\int_0^t \eta^x(u,t)\cdot dW_d^T(u), \int_0^s \eta^x(u,s)\cdot dW_d^T(u)\middle| Z\right)\\ &= \int_0^{\min\{s,t\}} \eta^x(u,t)\cdot\eta^x(u,s)du - m(t)m(s)\end{aligned}$$

The conditional expected value of the exchange rate is determined by

$$\begin{aligned}E_{P_d^T}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)}\middle| Z\right] &= E_{P_d^T}\left[F(0,t)\exp\left\{\int_0^t \eta^x(u,t)\cdot dW_d^T(u)\right\}\middle| Z\right]\\ &= F(0,t)\exp\left\{m(t)Z+\frac{1}{2}\nu^2(t,t)\right\}\end{aligned}$$

which obviously implies

$$E_{P_d^T}\left[E_{P_d^T}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)}\middle| Z\right]\right] = E_{P_d^T}\left[\frac{X(t)D^f(t,t)}{D^d(t,t)}\right]$$

and with (3.4) the lower bounds are determined. □

The formal difference between the exact equation for the value of an Asian option and the lower bound is that we are not integrating with respect to an n-dimensional Brownian motion. Applying an argument by Jamshidian (1991), a closed form solution for the lower bound of a fixed strike discretely sampled Asian option can now be derived.

To prepare for the main theorem of this paper we will reformulate the right hand side of the equation (3.6). Let

$$f_i(z) \quad := \quad \exp\left\{m_t(t_i)z + \frac{1}{2}\nu_t^2(t_i,t_i)\right\}$$

and observe that $f_i(z)$, for $m_t(t_i) > 0$, is an increasing convex function, and for $m_t(t_i) < 0$ a decreasing convex function. Observe also that the coefficient to $f_i(z)$ in the right hand side of (3.6), $F(t,t_i)$ is positive.

Denote by $\mathcal{P}_{n(t)}$ and $\mathcal{N}_{n(t)}$ the following sets

$$\begin{aligned} \mathcal{P}_{n(t)} &= \{i | m_t(t_i) > 0, i \in \{n(t)+1,...,N\}\} \\ \mathcal{N}_{n(t)} &= \{i | m_t(t_i) < 0, i \in \{n(t)+1,...,N\}\}. \end{aligned}$$

and let $\sum_{i\in\emptyset} F(t,t_i)f_i(z) := 0$ and $0\cdot\infty = 0$. With this notation the right hand side of (3.6) takes the form

$$D^d(t,T)E^t_{P_d^T}\left[\left[\frac{1}{N}\sum_{i\in\mathcal{P}_{n(t)}\cup\mathcal{N}_{n(t)}} F(t,t_i)f_i(Z) - K(t)\right]^+\right]$$

The equation

$$0 \quad = \quad \frac{1}{N}\sum_{i\in\mathcal{P}_{n(t)}\cup\mathcal{N}_{n(t)}} F(t,t_i)f_i(z) - K(t)$$

has zero, one or two solutions.

Definition 3.1:

• $\mathcal{P}_{n(t)} \neq \emptyset$, $\mathcal{N}_{n(t)} \neq \emptyset$ *and* $\frac{1}{N} \sum_{i \in \mathcal{P}_{n(t)} \cup \mathcal{N}_{n(t)}} F(t,t_i) f_i(z) - K(t) < 0$ *for some value of* z.

Denote the two unique solutions by z^* *and* z^{**} *respectively and choose* $z^* < z^{**}$, *and define*

$$K_i^*(t) := F(t,t_i) f_i(z^*)$$

$$K_i^{**}(t) := F(t,t_i) f_i(z^{**}) \quad .$$

• $\mathcal{P}_{n(t)} \neq \emptyset$, $\mathcal{N}_{n(t)} \neq \emptyset$ *and* $\frac{1}{N} \sum_{i \in \mathcal{P}_{n(t)} \cup \mathcal{N}_{n(t)}} F(t,t_i) f_i(z) - K(t) \geq 0 \; \forall z$.

Define $z^* = z^{**} := \infty$ *and* $K_i^*(t) = K_i^{**}(t) := K(t)$

• $\mathcal{P}_{n(t)} \neq \emptyset$ *but* $\mathcal{N}_{n(t)} = \emptyset$.

Denote the unique solution by z^{**} *and define* $z^* := -\infty$, $K_i^*(t) := K(t)$ *and*

$$K_i^{**}(t) := F(t,t_i) f_i(z^{**}) \quad .$$

• $\mathcal{P}_{n(t)} = \emptyset$ *but* $\mathcal{N}_{n(t)} \neq \emptyset$.

Denote the unique solution by z^* *and define* $z^{**} := \infty$, $K_i^{**}(t) := K(t)$ *and*

$$K_i^*(t) := F(t,t_i) f_i(z^*) \quad .$$

With these preparations in place we obtain that

$$\begin{aligned}
(3.8) \qquad & \left(\frac{1}{N} \sum_{i \in \mathcal{P}_{n(t)} \cup \mathcal{N}_{n(t)}} F(t,t_i) f_i(z) - K(t) \right)^+ \\
= \; & \frac{1}{N} \sum_{i \in \mathcal{N}_{n(t)}} \left(F(t,t_i) f_i(z) - K_i^*(t) \right)^+ - \frac{1}{N} \sum_{i \in \mathcal{P}_{n(t)}} \left(K_i^*(t) - F(t,t_i) f_i(z) \right)^+ \\
& + \frac{1}{N} \sum_{i \in \mathcal{P}_{n(t)}} \left(F(t,t_i) f_i(z) - K_i^{**}(t) \right)^+ - \frac{1}{N} \sum_{i \in \mathcal{N}_{n(t)}} \left(K_i^{**}(t) - F(t,t_i) f_i(z) \right)^+
\end{aligned}$$

Through the conditional expectation approach the original Asian option problem was turned into a formulation where we should find the price of an option written on a portfolio with the value at time t_N given by $\frac{1}{N} \sum_{i \in \mathcal{P}_{n(t)} \cup \mathcal{N}_{n(t)}} F(t,t_i) f_i(z)$. Through the reformulation just performed this has been tranformed to the problem where the price of a portfolio of options should be found. The reformulation is an extension of the arguments found in Jamshidian (1991). Each individual option in the portfolio is of the simple Blach–Scholes type.

Theorem 3.1:

Under the Assumptions of Proposition 3.1 the lower bound of a fixed strike discretely sampled Asian call option at time t with $K(t) > 0$ is equal to

$$D^d(t,T)\frac{1}{N}\sum_{i=n(t)+1}^{N}\left(E^t_{P_d^T}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right]N\left(z^* - m_t(t_i)\right) - K_i(t)N(z^*)\right) \tag{3.9}$$
$$+D^d(t,T)\frac{1}{N}\sum_{i=n(t)+1}^{N}\left(E^t_{P_d^T}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right]N\left(-z^{**} + m_t(t_i)\right) - K_i(t)N(-z^{**})\right)$$

where $N(\cdot)$ denotes the cumulative standard normal distribution, z^*, z^{**}, $K_i^*(t)$ and $K_i^{**}(t)$ are given in the Definition 3.1.

Proof. To simplify the proof it is sufficient to consider the situation at time zero. The lower bound of the fixed strike discretely sampled Asian call option is equal to

$$D^d(0,T)E_{P_d^T}\left[\left[\frac{1}{N}\sum_{i=1}^{N}F(0,t_i)\exp\left\{m(t_i)Z + \frac{1}{2}\nu^2(t_i,t_i)\right\} - K\right]^+\right]$$

Applying 3.8 and denoting the density function for Z by $h(z)$ we obtain immediately

$$D^d(0,T)E_{P_d^T}\left[\left[\frac{1}{N}\sum_{i=1}^{N}F(0,t_i)\exp\left\{m(t_i)Z + \frac{1}{2}\nu^2(t_i,t_i)\right\} - K\right]^+\right]$$
$$= \frac{D^d(0,T)}{N}\Bigg(\sum_{i\in\mathcal{N}_{n(t)}}\int_{-\infty}^{z^*}\left(F(t,t_i)f_i(z) - K_i^*(t)\right)h(z)dz - \sum_{i\in\mathcal{P}_{n(t)}}\int_{-\infty}^{z^*}\left(K_i^*(t) - F(t,t_i)f_i(z)\right)h(z)dz$$
$$+\sum_{i\in\mathcal{P}_{n(t)}}\int_{z^{**}}^{\infty}\left(F(t,t_i)f_i(z) - K_i^{**}(t)\right)h(z)dz - \sum_{i\in\mathcal{N}_{n(t)}}\int_{z^{**}}^{\infty}\left(K_i^{**}(t) - F(t,t_i)f_i(z)\right)h(z)dz\Bigg)$$

Using that Z is a standard normally distributed variable and that by definition

$$F(0,t_i)\exp\left\{\frac{1}{2}\nu^2(t_i,t_i) + \frac{1}{2}m(t_i)^2\right\}$$
$$= \frac{X(t_i)D^f(0,t_i)}{D^d(0,t_i)}\exp\left\{-\int_0^{t_i}\eta^x(u,t_i)\eta^d(u,t_i,T)du\right\}$$
$$= E_{P_d^T}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right]$$

we obtain the expression stated in the Theorem. □

The lower bound for a fixed strike discrete Asian put option is equal to

$$
\begin{aligned}
(3.10) \qquad & D^d(t,T)\frac{1}{N}\sum_{i=n(t)+1}^{N}\left(K_i(t)N(-z^*) - E^t_{P_d^T}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right]N\left(-z^* + m_t(t_i)\right)\right) \\
& +D^d(t,T)\frac{1}{N}\sum_{i=n(t)+1}^{N}\left(K_i(t)N(z^{**}) - E^t_{P_d^T}\left[\frac{X(t_i)D^f(t_i,t_i)}{D^d(t_i,t_i)}\right]N\left(z^{**} - m_t(t_i)\right)\right)
\end{aligned}
$$

Applying the change of measure to P_X, similar closed form solutions can be derived to approximate the floating strike Asian option.

So far the closed form solution of the lower bound depends on the specific choice of the conditioning random variable Z. As Rogers and Shi (1995), we fix Z such that the conditional variance of the terminal payoff is small. In order to calculate the conditional variance at time t note that

$$
\begin{aligned}
& \operatorname{cov}^t_{P_d^T}\left(\exp\left\{\int_t^{\tau}\eta^x(u,\tau)\cdot dW_d^T(u)\right\},\exp\left\{\int_t^{s}\eta^x(u,s)\cdot dW_d^T(u)\right\}\middle| Z\right) \\
= \; & \exp\left\{(m_t(\tau)+m_t(s))Z + \frac{1}{2}(\nu_t^2(\tau,\tau)+\nu^2(s,s))\right\}\left(\exp\{\nu_t^2(s,\tau)\}-1\right)
\end{aligned}
$$

The conditional variance for the discrete Asian option at time t is equal to

$$
\begin{aligned}
& V^t_{P_d^T}[A_d(T)|Z] \\
= \; & \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\sum_{j=n(t)+1}^{N}F(t,t_i)F(t,t_j) \\
& \exp\left\{(m_t(t_i)+m_t(t_j))Z + \frac{1}{2}(\nu_t^2(t_i,t_i)+\nu_t^2(t_j,t_j))\right\}\left(\exp\{\nu_t^2(t_i,t_j)\}-1\right)
\end{aligned}
$$

and for the continuous Asian option

$$
\begin{aligned}
& V^t_{P_d^T}[A_c(T)|Z] \\
= \; & \frac{1}{T^2}\int_t^T\left(\int_t^T F(t,u)F(t,s)\exp\left\{(m_t(u)+m_t(s))Z+\frac{1}{2}(\nu_t^2(u,u)+\nu_t^2(s,s))\right\}\left(\exp\{\nu_t^2(u,s)\}-1\right)ds\right)du
\end{aligned}
$$

Using the approximation $\exp\{x\} \approx 1 + x$ and the fact $\nu_t^2(s,t) = \nu_t^2(t,s)$ the conditional variance can be approximated by

$$
\begin{aligned}
& V^t_{P_d^T}[A_d(T)|Z] \\
\approx\ & \frac{1}{N^2} \sum_{i=n(t)+1}^{N} \sum_{j=n(t)+1}^{N} \Bigg(X(t)^2 \frac{D^f(t,t_i)}{D^d(t,t_i)} \frac{D^f(t,t_j)}{D^d(t,t_j)} \nu_t^2(t_i,t_j) \\
& \left[1 - \frac{1}{2} \int_t^{t_i} \eta^x(u,t_i) \cdot (\eta^x(u,t_i) + 2\eta^d(u,t_i,T)) du + m_t(t_i) Z + \frac{1}{2} \nu_t^2(t_i,t_i) \right. \\
& \left. - \frac{1}{2} \int_t^{t_j} \eta^x(u,t_j) \cdot (\eta^x(u,t_j) + 2\eta^d(u,t_j,T)) du + m_t(t_j) Z + \frac{1}{2} \nu_t^2(t_j,t_j) \right] \Bigg) \\
=\ & \frac{1}{N^2} \sum_{i=n(t)+1}^{N} \left(X(t)^2 \frac{D^f(t,t_i)}{D^d(t,t_i)} \left[1 - \int_t^{t_i} \left(\eta^x(u,t_i) \cdot (\eta^x(u,t_i) + 2\eta^d(u,t_i,T)) \right) du + 2m_t(t_i) Z + \nu_t^2(t_i,t_i) \right] \right. \\
& \left. \cdot \left(\sum_{j=n(t)+1}^{N} \nu_t^2(t_i,t_j) \frac{D^f(t,t_j)}{D^d(t,t_j)} \right) \right)
\end{aligned}
$$

Since by definition

$$
\sum_{j=N(t)+1}^{N} \nu_t^2(t_i,t_j) \frac{D^f(t,t_j)}{D^d(t,t_j)} = \operatorname{cov}^t_{P_d^T} \left(\int_t^{t_i} \eta^x(u,t_i) \cdot dW_d^T(u), \sum_{j=n(t)+1}^{N} \frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{t_j} \eta^x(u,t_j) \cdot dW_d^T(u) \middle| Z \right),
$$

the conditional covariance at time t is approximated by zero if we set the random variable Z equal to

$$
\begin{aligned}
(3.11) \qquad Z &= \frac{1}{\alpha_t} \left(\sum_{j=n(t)+1}^{N} \frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{t_j} \eta^x(u,t_j) \cdot dW_d^T(u) \right) \\
\alpha_t^2 &= V^t_{P_d^T} \left[\sum_{j=n(t)+1}^{N} \frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{t_j} \eta^x(u,t_j) \cdot dW_d^T(u) \right] \\
&= \sum_{k=1}^{n} \left(V^t_{P_d^T} \left[\sum_{j=n(t)}^{N-1} \int_{\max\{t,t_j\}}^{t_{j+1}} \left(\sum_{i=j+1}^{N} \frac{D^f(t,t_i)}{D^d(t,t_i)} \eta_k^x(u,t_i) \right) \cdot dW_{d,k}^T(u) \right] \right) \\
&= \sum_{k=1}^{n} \left(\sum_{j=n(t)}^{N-1} \left(\int_{\max\{t,t_j\}}^{t_{j+1}} \left(\sum_{i=j+1}^{N} \frac{D^f(t,t_i)}{D^d(t,t_i)} \eta_k^x(u,t_i) \right)^2 du \right) \right) \\
\Longrightarrow \quad m_t(t_i) &= E^t_{P_d^T} \left[Z \int_t^{t_i} \eta_k^x(u,t_i) \cdot dW_d^T(u) \right] \\
&= \frac{1}{\alpha_t} \sum_{j=n(t)+1}^{N} \left[\frac{D^f(t,t_j)}{D^d(t,t_j)} \int_t^{\min\{t_j,t_i\}} \eta^x(u,t_i) \cdot \eta^x(u,t_n) du \right]
\end{aligned}
$$

Alternatively we could use the approximation

$$\begin{aligned} V^t_{P^T_d}[A_d(T)|Z] &\approx \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\sum_{j=n(t)+1}^{N} X^2(t)\nu_t^2(t_i,t_j) \\ &\quad \left[1+\ln F(t,t_i)+m_t(t_i)Z+\frac{1}{2}\nu_t^2(t_i,t_i)+\ln F(t,t_j)+m_t(t_j)Z+\frac{1}{2}\nu_t^2(t_j,t_j)\right] \\ &= \frac{1}{N^2}\sum_{i=n(t)+1}^{N}\left(X^2(t)\left[1+2\ln F(t,t_i)+2m_t(t_i)Z+\nu_t^2(t_i,t_i)\right]\sum_{j=n(t)+1}^{N}\nu_t^2(t_i,t_j)\right) \end{aligned}$$

This suggests a specification of the random variable Z equal to

$$\begin{aligned} Z &= \frac{1}{\alpha_t}\left(\sum_{j=n(t)+1}^{N}\int_t^{t_j}\eta^x(u,t_j)\cdot dW_d^T(u)\right) \\ \alpha_t^2 &= \sum_{k=1}^{n}\left(\sum_{j=n(t)}^{N-1}\left(\int_{\max\{t,t_j\}}^{t_j+1}\left(\sum_{i=j+1}^{N}\eta_k^x(u,t_i)\right)^2 du\right)\right) \\ \Longrightarrow \quad m_t(t_i) &= \frac{1}{\alpha_t}\sum_{j=n(t)+1}^{N}\left[\int_t^{\min\{t_j,t_i\}}\eta^x(u,t_i)\cdot\eta^x(u,t_j)du\right] \end{aligned} \tag{3.12}$$

In case of a continuously sampled Asian option, the same argument implies that the conditional covariance is approximated by zero if we choose Z equal to

$$\begin{aligned} Z &= \frac{1}{\alpha_t}\int_t^T\left(\int_t^s\eta^x(u,s)\cdot dW_d^T(u)\right)\frac{D^f(t,s)}{D^d(t,s)}ds \\ \alpha_t^2 &= \sum_{k=1}^{n}\left[\int_t^T\left(\int_u^T\eta_k^x(u,s)\frac{D^f(t,s)}{D^d(t,s)}ds\right)^2 du\right] \\ \Longrightarrow \quad m_t(\tau) &= \frac{1}{\alpha_t}\int_t^T\frac{D^f(t,s)}{D^d(t,s)}\left(\int_t^{\min\{\tau,s\}}\eta^x(u,s)\cdot\eta^x(u,\tau)du\right)ds \quad . \end{aligned} \tag{3.13}$$

Using the alternative approximation the Z vector is found to be

$$\begin{aligned} Z &= \frac{1}{\alpha_t}\int_t^T\left(\int_t^s\eta^x(u,s)\cdot dW_d^T(u)\right)ds \\ \alpha_t^2 &= \sum_{k=1}^{n}\left[\int_t^T\left(\int_u^T\eta_k^x(u,s)ds\right)^2 du\right] \\ \Longrightarrow \quad m_t(\tau) &= \frac{1}{\alpha_t}\int_t^T\left(\int_t^{\min\{\tau,s\}}\eta^x(u,s)\cdot\eta^x(u,\tau)du\right)ds \quad . \end{aligned} \tag{3.14}$$

The approximation error to the price of an exchange rate Asian option is now determined by equation (3.5). For a discrete Asian option the approximation error is estimated by:

$$
\begin{aligned}
& D^d(t,T)\,\varepsilon_{RS} \\
= \quad & \frac{D^d(t,T)X(t)}{N}\left[\sum_{i=n(t)+1}^{N}\ \sum_{j=n(t)+1}^{N}\left(\frac{D^f(t,t_i)}{D^d(t,t_i)}\frac{D^f(t,t_j)}{D^d(t,t_j)}\ \exp\{m_t(t_i)m_t(t_j)\}\right.\right. \\
& \left.\left.\exp\left\{-\frac{1}{2}\int_t^{t_i}\eta^x(u,t_i)\cdot\eta^d(u,t_i,T)du-\frac{1}{2}\int_t^{t_j}\eta^x(u,t_j)\cdot\eta^d(u,t_j,T)du\right\}\left(\exp\left\{\nu_t^2(t_i,t_j)\right\}-1\right)\right)\right]^{\frac{1}{2}},
\end{aligned}
$$

and in the case of a continuous Asian option by:

$$
\begin{aligned}
& D^d(t,T)\,\varepsilon_{RS} \\
= \quad & \frac{D^d(t,T)X(t)}{T}\left[\int_t^T\left(\int_t^T\ \frac{D^f(t,t_i)}{D^d(t,t_i)}\frac{D^f(t,t_j)}{D^d(t,t_j)}\ \exp\{m_t(t_i)m_t(t_j)\}\right.\right. \\
& \left.\left.\exp\left\{-\frac{1}{2}\int_t^{u}\eta^x(v,u)\cdot\eta^d(v,u,T)dv-\frac{1}{2}\int_t^{s}\eta^x(v,s)\cdot\eta^d(v,s,T)dv\right\}\left(\exp\left\{\nu_t^2(u,s)\right\}-1\right)du\right)ds\right]^{\frac{1}{2}}.
\end{aligned}
$$

4. Conclusion

Foreign exchange rate Asian options have been analysed with the primary aim to find a good approximation for their pricing. A model for the foreign exchange should describe not only the exchange rate itself but also the term structure of interest rates in the two countries. The total correlation structure in this two-country economy will be important for the pricing purpose.

The Asian option is in this paper priced through the Rogers and Shi (1995) approach combined with an exact pricing of their approximated lower bound. This exact pricing takes the form of a sum of Black-Scholes options. The pricing error for the Rogers and Shi approximation is developed.

Furthermore, we plan to apply the methodology to other financial contracts with a similar mathematical structure. Examples are basket options and more generally n-color rainbow type options. As in our case the methodology may imply an approximation to the value of these options by a portfolio of simpler financial contracts.

It is ongoing research to cover the important topic of hedging long term Asian options in an international setting.

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