On the Nature of Options

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Initial version: April 19, 2000

File reference: nature5.tex

We thank Claudio Albanese, Ali Hirsa, Tom Little, Vijay Pant, and Jeremy Staum for their comments. We also thank the participants of presentations at Columbia University, Global Derivatives 2000, and ITAM.

Abstract

We consider the role of options when markets in its underlying asset are frictionless and when this underlying has a volatility process and jump arrival rates which are arbitrarily stochastic. By combining a static option position with a particular dynamic hedging strategy, we characterize the option's time value as the (risk-neutral) expected benefit from being able to buy or sell one share of the underlying at the option's strike whenever the strike price is crossed. The buy/sell decision can be based on the post jump price, so that a rational investor buys on rises and sells on drops. Thus, an option provides liquidity at its strike even when the market doesn't. We next present two methods for extending this local liquidity to every price between the pre and post jump level. The first method involves holding a continuum of options of all strikes. The second method holds one option, but adjusts the dynamic hedging strategy. We discuss the advantages and disadvantages of each approach and consider the benefits of combining them.

Introduction

The standard theory of contingent claim pricing through dynamic replication gives no special role to options. Using Monte Carlo simulation, path-dependent multivariate claims of great complexity can be priced as easily as the path-independent univariate hockey-stick payoffs which characterize options. It is thus not at all obvious why markets have organized to offer these simple payoffs, when other collections of functions such as polynomials, circular functions, or wavelets might offer greater advantages.

While the complete answer to this question is undoubtedly highly complex, the purpose of this paper is to analyze the particular role which options play when markets for their underlying are frictionless, and when the underlying stock has arbitrarily stochastic volatility and jumps with stochastic arrival rates. In this context, we show how options can be regarded as vehicles for enhancing dynamic trading in their underlying assets by allowing implicit trading in the underlying at one or more price levels, even if the market jumps across the level.

Aside from highlighting the trading role of options, our analysis has implications for both risk management and product innovation. On the risk management side, we derive a general expression which shows the sources of hedging error which arise when a portfolio of options is incorrectly hedged by assuming that the volatility and/or arrival rate (as given by the Lévy measure) is constant, when either or both quantities is stochastic. On the product innovation side, we show how such errors can be turned into opportunities by presenting several special cases of hedging error which might be sold off into the market as an attractive contingent claim. In contrast to most derivative securities, the payoff to these claims can be perfectly replicated, even though markets are incomplete due to arbitrarily stochastic volatility and jump sizes.

The remainder of this paper is organized as follows. The next section considers the cash flow which arises by combining a static position in a single option with a particular dynamic hedging strategy. We show that an option's time value can be characterized as the premium paid to be able to buy or sell at most one unit of the underlying on every cross of the strike. The transaction price is fixed at the strike, while the volume which can be bought or sold can be based on the post jump price. As a result, every rational investor will buy one unit of the underlying at the option's strike on every up jump crossing the strike from below, and sell one unit at the option's strike on every drop crossing the strike from above. The next three sections present two different methods for extending this local liquidity to a continuum of prices. The first method involves holding a continuum of options of all strikes. The second method holds one option, but adjusts the dynamic hedging strategy. The penultimate section discuss the advantages and disadvantages of each approach, and considers the benefit of combining them. The final section summarizes the paper and discusses various extensions. The appendices collect various technical derivations.

I A Characterization of Option Time Value

This section characterizes the time value of a standard call option of maturity T and strike K. By put call parity, the characterization also applies to the time value of the corresponding put. We assume that an investor has the ability to open forward contracts in the underlying with delivery price K and maturity T and to trade pure discount bonds paying one dollar at T (henceforth the terms "forward contract" and "bond" refer to these particular instruments). We further assume that the forward price process $\{F_t, t \in [0,T]\}$ is a continuous time semi-martingale[1] under the statistical probability measure P. To prevent negative prices, we assume that the forward price cannot jump across the origin, and that the origin is an absorbing state.

Consider the binary strategy of holding nothing for all forward prices at or below K and being short one forward contract otherwise[2]. This strategy involves dynamic trading only when the forward price touches or crosses the strike K. If the forward price is below the strike at some time $t \geq 0$ and then rises across, then a short forward position is opened at the post jump price F_t. Conversely, if the forward price is above the strike at some time $t > 0$ and then drops across, then an offsetting long forward position is opened at the post jump price F_t. Since *limit orders*[3] can be used to ensure that the positions

[1]Semi-martingales are the widest class of stochastic processes for which arbitrage pricing applies.

[2]The negative of this strategy was first considered in Seidenverg[7] and analyzed further in Carr and Jarrow[1].

[3]Limit orders turn into market orders as soon as a barrier is touched or crossed. Sell short limit orders would be used to open a short forward position on a rise above the barrier and buy limit orders would be used to offset it on a drop below.

are opened at the first touch or cross of the barrier, we term the strategy a limit order strategy. The changes in the forward contract position generate cash flows at strike crossings, which are discussed in more detail below. Temporarily ignoring these cash flows, the strategy results in a liquidating payoff of $-1(F_T > K)(F_T - K) = -(S_T - K)^+$ at T. To eliminate this liability, suppose that an investor also pays C_0 initially to buy a standard European call struck at $K \geq 0$, and then holds the call to its maturity date T. Since the limit order strategy is now enhanced by the addition of a static call position, the combined strategy is termed an *enhanced* limit order strategy. After paying the initial premium and forming the initial forward position, if any, this strategy only generates nonnegative receipts for the investor. These cash inflows arise because the forward contracts struck at K are only sold when the forward price is above K and purchased when the forward price is at K or below. Appendix 2 proves a generalization of the following theorem, which tallies the profit and loss (P&L) from this strategy:

Theorem 1: *Assuming frictionless markets in a call's underlying asset and a semi-martingale process for the forward price, the terminal P&L from the enhanced limit order strategy is:*

$$\begin{aligned} P\&L_T &= -[C_0 e^{rT} - (F_{0T} - K)^+] + \frac{1}{2} L_T^F(K) \\ &+ \sum_{0<t\leq T} \left[1(F_{t-} \leq K)(F_t - K)^+ + 1(F_{t-} > K)(K - F_t)^+ \right], \end{aligned} \tag{1}$$

where $L_T^F(K)$ is the terminal local time of the forward price at the strike.

Loosely speaking, local time is the part of the terminal quadratic variation (normal variance) which is experienced when the forward price is near the strike. To interpret (1), we initially assume that the forward price process has no diffusion component, so that the local time vanishes. The quantity $1(F_{t-} \leq K)(F_t - K)^+$ in the last term in (1) is only non-zero when the forward price jumps up across K at t. In this case, this term is the instantaneous profit realized at t by going long a costless forward contract when the forward price is at K and then liquidating it at $F_t > K$. Similarly, the quantity $1(F_{t-} > K)(K - F_t)^+$ in the last term in (1) is only non-zero when the forward price drops across K at t. In this case, this term is the instantaneous profit realized at t by[4] going short a costless forward contract when the forward price is at K and then liquidating it at $F_t < K$. Thus, (1) indicates that by initially paying the call's

[4]The profits realized at every $t \in [0, T]$ are deferred without interest to T.

time value $C_0 - (F_{0T} - K)^+ e^{-rT}$, the investor inherits the ability to go long or short at most one forward contract every time that the strike price is crossed. The delivery price of the zero cost forward position is fixed at K, while the volume decision can be conditioned on the level of the post jump forward price. Hence, any rational investor would choose to go long one forward contract every time that the forward price crosses K from below, and short a forward contract on every cross from above. Thus, the purchase of an out-of-the-money option provides liquidity in the forward at its strike, even when the market doesn't[5].

If the price process has a diffusion component, then the local time term arises. The origin of this term is exactly the same as in the pure jump case, i.e. the ability to choose a long position on price rises and a short position on price drops. It is well known that this local time term arises purely due to activity of the forward near the strike. In the diffusion case, the small magnitude of each price change is offset by the infinite frequency with which they occur. The outcome of an infinite series of infinitessimal profits is a random process contributing to the P&L, which increases only when the forward is at the strike.

I-A Breakout Option

The cash flows generated by the enhanced limit order strategy can be packaged into an attractive product for investors. This "breakout option" pays the straight sum of all the "overshoots" of a fixed level during its life. Under deterministic interest rates, the product can be alternatively written on a futures price, if this is more transparent. It is also straightforward to have discrete (eg. daily) monitoring, in which case the hedge involves trading in the underlying at most daily. By purchasing a standard call struck at the overshoot level and with the same life as the product, and undertaking the limit order strategy, the payoff can be replicated perfectly, even though markets may be grossly incomplete due to arbitrarily stochastic volatility and arrival rates. Thus, the arbitrage-free price of this breakout option is the initial time value of a standard option. By adding the call to the limit order strategy, the hedger inherits a localized liquidity which is unavailable to other market participants. This access to markets allows the investor to perfectly replicate the payoffs to the contract. We next consider two alternative approaches for extending this local

[5]This insight is first expounded by Seidenverg[7] in a binomial context. It is extended to continuous price processes in Carr and Jarrow[1] and to arbitrary semi-martingales in this paper.

liquidity to a continuum of strikes. This extension will also generate some other product ideas.

II Positioning in a Continuum of Strikes

In this section, we consider static positions in options of the same maturity T. We allow for long or short positions in options with a continuum of strikes $K > 0$, and for infinitessimal or discrete positions in each option. In this market, it is well known that an investor can create an arbitrary payoff at T, which is some function[6] $f(\cdot)$ of the final spot price S_T. Indeed, Appendix 4 proves the following version of Taylor series:

$$f(F) = f(\kappa) + f'(\kappa)(F - \kappa) + \int_{\kappa+}^{\infty} f''(K)(F - K)^+ dK + \int_{0}^{\kappa-} f''(K)(K - F)^+ dK, \tag{2}$$

where $\kappa \geq 0$ is any expansion point. Thus, the payoff decomposes into bonds, forward contracts with delivery price κ, calls struck above κ, and puts struck below. Letting V_0^f denote the initial value of the payoff $f(S_T)$ at T, the absence of arbitrage implies:

$$V_0^f = f(\kappa)B_0 + f'(\kappa)(F_0 - \kappa)B_0 + \int_{\kappa+}^{\infty} f''(K)C_0(K)dK + \int_{0}^{\kappa-} f''(K)P_0(K)dK, \tag{3}$$

where B_0, C_0, and P_0 denote the initial prices of bonds, calls, and puts respectively.

In the last section, the number of forward contracts held at t was simply the negative of the left derivative of $f(F_t) = (F_t - K)^+$ w.r.t. F_t. If the same rule is used for an arbitrary payoff function f, then Appendix 2 proves that a generalization of the following theorem holds:

Theorem 2: *Assume that we have frictionless markets in the asset underlying a continuum of options of maturity T. Also assume a semi-martingale process for the forward price. Consider the terminal P&L from combining dynamic trading in forward contracts with a static position in bonds, forward contracts, and options which pays off $f(S_T)$ at T. Suppose that the static position is purchased for V_0^f initially, and that the number of forwards held over $[0, T)$ is the negative of the left derivative of $f(F_t)$ w.r.t. F_t. Then*

[6] Richardson extrapolation can be used to improve the approximation of the payoff function $f(\cdot)$ when strikes are discrete.

the terminal P&L is:

$$P\&L_T = -\left[\frac{V_0^f}{B_0} - f(F_0)\right] + \frac{1}{2}\int_0^\infty L_T^F(K)m(dK) + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_t} f''(K)(F_t - K)dK\mu(dx,dt), \qquad (4)$$

where m is a signed measure giving the second derivative of f in the generalized function sense, and $\mu(dx, dt)$ is the integer valued random measure which counts the number of jumps in any region of space-time.

The first term $\frac{V_0^f}{B_0} - f(F_0)$ in (4) is the (forward value of the) time value of an arbitrary claim with payoff $f(F)$ at T. In return for paying this amount, the investor receives a linear combination of the local times across strikes, as well as the last term in (4). The inner integral in the last term in (4) is the instantaneous P&L which would arise at t for an investor with unusually fine access to markets and the uncanny ability to buy on rises and sell on drops. Specifically, as the forward price at t jumps up from F_{t-} to F_t, this investor is able to go long $f''(K)dK$ costless forward contracts at each price K inbetween the two prices. If the long position in the array of forward contracts is reversed at the forward price F_t, then the instantaneous P&L is this inner integral $\int_{F_{t-}}^{F_t} f''(K)(F_t - K)dK$. Similarly, if the forward price jumps down from F_{t-} to F_t, then the integral can be written as $\int_{F_t}^{F_{t-}} f''(K)(K - F_t)dK$, indicating that the investor is able to go short $f''(K)dK$ costless forward contracts at each price K inbetween the pre and post-jump prices.

Thus, the purchase of options with a continuum of strikes provides liquidity at every strike price. If $f''(K) > 0$ at K, then the investor also inherits the ability to buy on rises and sell on drops, whereas if $f''(K) < 0$ at K, then the investor sells this ability. For future use, we refer to $f''(K)$ as the local liquidity rate at price K. Thus, a portfolio of options of several strikes provides non-zero local liquidity at each strike before they mature, so long as each option is coupled with a limit order strategy in the underlying.

In the special case where the investor holds one call struck at K_c, (i.e. $f(F) = (F - K_c)^+$), then the local liquidity rate at each future time $t \in [0, T]$ and price level $K > 0$ is $f''(K)dK = \delta(K - K_c)$, where $\delta(\cdot)$ is a Dirac delta function[7]. Thus, at each $t \in [0, T]$, the investor trades one contract if the price crosses K_c then, and doesn't trade otherwise.

[7] The Dirac delta function is a generalized function characterized by two properties:

II-A Price Variance Swap

The result in Theorem 2 can be used as the basis for synthesizing another interesting financial product. Suppose that the investor chooses to create the payoff $f(S) = S^2$ at T. Choosing $\kappa = F_0$ in the Taylor series described in (2),

$$F_T^2 = F_0^2 + 2F_0(F_T - F_0) + \int_{F_0+}^{\infty} 2(F_T - K)^+ dK + \int_{0}^{F_0-} 2(K - F_T)^+ dK. \tag{5}$$

Thus, to create the quadratic payoff F_T^2, the investor initially buys F_0^2 bonds, $2F_0$ costless forward contracts, $2dK$ of each of the out-of-the-money calls, and $2dK$ of each of the out-of-the-money puts. From (3), the initial cost of creating this payoff is:

$$V_0^{S^2} = F_0^2 B_0 + \int_{F_0+}^{\infty} 2C_0(K)dK + \int_{0}^{F_0-} 2P_0(K)dK. \tag{6}$$

From Theorem 2, the terminal P&L from layering on a dynamic position in $-2F_t$ forward contracts is:

$$P\&L_T = F_0^2 - \frac{V_0^{S^2}}{B_0} + \int_0^\infty L_T^F(K)dK + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_t} 2(F_t - K)dK\mu(dx, dt), \tag{7}$$

From (6) and the properties of local time, this simplifies to:

$$P\&L_T = -\int_{F_0+}^{\infty} 2\frac{C_0(K)}{B_0}dK - \int_{0}^{F_0-} 2\frac{P_0(K)}{B_0}dK + [F,F]_T^c + \int_0^T \int_{-\infty}^{\infty} (F_t - F_{t-})^2 \mu(dx, dt), \tag{8}$$

where $[F,F]_T^c$ is the terminal quadratic variation from the continuous component of the forward price process. Suppose that our product has a nonnegative payoff at maturity equal to the last two terms in (8), which are the sum of the squared price changes. If the premium for this payoff is paid at maturity then from (8), the fair fixed price to charge for this random payoff is $\int_{F_0+}^{\infty} 2\frac{C_0(K)}{B_0}dK + \int_{0}^{F_0-} 2\frac{P_0(K)}{B_0}dK$. To express this fixed payment as a rate per unit time, define:

$$\bar{v}^2 \equiv \frac{\int_{F_0+}^{\infty} 2\frac{C_0(K)}{B_0}dK + \int_{0}^{F_0-} 2\frac{P_0(K)}{B_0}dK}{T}.$$

1. $\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$
2. $\int_{-\infty}^{\infty} \delta(x)dx = 1.$

.

Then, the terminal P&L in (8) is that of a price variance swap:

$$P\&L_T = [F,F]^c_T + \int_0^T \int_{-\infty}^{\infty} (F_t - F_{t-})^2 \mu(dx,dt) - \int_0^T \bar{v}^2 dt. \quad (9)$$

In addition to the options, the hedger of a short price variance swap holds F_0^2 bonds and $-2(F_t - F_0)$ forward contracts at each time $t \in [0,T]$. It is straightforward to show that discretely monitored price variance swaps can also be replicated perfectly. Perhaps surprisingly, the fair price is independent of the monitoring frequency. Furthermore, the dynamic component of the forward position is only adjusted on each monitoring date[8].

Under further assumptions, price variance swaps can be used to obtain a forecast of future price variance rates. If a continuum of option maturities also trade at time 0, then calendar spreads of price variance swaps can be used to infer the term structure of forward variance rates. Since this forecast is risk-neutral, an adjustment for risk aversion is required to obtain a forecast under the statistical measure.

As in the Heath-Jarrow-Morton[5] paradigm (henceforth HJM), this term structure of forward variance rates can be used as the basis for a theory of purely dynamic replication. By assuming some dynamics for the evolution of this term structure (eg. the term structure is a diffusion driven by a single Brownian motion), the dimension of the replication problem is dramatically reduced. In contrast to the standard HJM model, the risk-neutral drift of these forward variance rates is simply zero (under the forward measure). In the one dimensional[9] diffusion case, this bypasses the issue of explosions (see eg. Karatzas and Shreve[6] pg. 332) allowing greater freedom in the selection of a diffusion coefficient. See Dupire[3] for a similar approach when the underlying price process has no jumps.

[8]As a further variation on a price variance swap, consider the continuous cash flow resulting from holding $2dK$ of each of the initially out-of-the-money options with strikes in the interval $[L,H]$, where $L < F_0 < H$, and then delta-hedging using the negative of the first derivative of the payoff. It is straightforward to show that this continuous cash flow is $(F_t - \underline{\bar{F}}_{t-})^2 - (F_t - \underline{\bar{F}}_t)^2$, where $\underline{\bar{F}} \equiv \max[L, \min(F,H)]$ is the forward price floored at L and capped at H.

[9]Explosions can arise with zero drift in the multi-dimensional case. Therefore, to avoid explosions when evolving a term structure of price variance swap rates, it is sufficient that each rate's volatility depend on at most that rate and time.

III Positioning in a Continuum of Strikes and Maturities

III-A Achieving Desired Local Liquidity

We have seen that by holding $f''(K)dK$ of each strike of fixed maturity T, the investor can generate the local liquidity rate $f''(K)$ for each $t \in [0, T]$. Ignoring the local time term for simplicity, this leads to a random component of time T P&L given by the last term of (4):

$$\int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_t} \frac{\partial^2 f}{\partial K^2}(K, T)(F_t - K)dK\mu(dx, dt), \tag{10}$$

where the dependence of the payoff f on its maturity T has been made explicit. Suppose more generally that an investor wishes to achieve a certain local liquidity rate $\ell(K, t)$ which varies over time as well as across prices. In other words, the investor wishes to have the term

$$\int_0^\tau \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_t} \ell(K, t)(F_t - K)dK\mu(dx, dt), \tag{11}$$

paid to him at some fixed time $\tau > 0$, where the maturity of the forward price is now τ. This subsection shows that by coupling options of all maturities $T \in [0, \tau]$ and strikes $K > 0$ with a limit order strategy in the underlying, the investor can generate an arbitrary given local liquidity rate $\ell(K, t)$ for $K > 0$ and $t \in [0, \tau]$.

For simplicity, suppose that options of all maturities $T \in [0, \tau]$ are now written on the forward price with maturity τ. Further suppose that the option price negotiated upon at time 0 is actually paid at τ. All dynamic trading is now in the τ maturity forward contract with delivery price K. Integrating (10) over T from 0 to τ and employing Fubini's theorem yields:

$$\int_0^\tau \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_t} \int_t^\tau \frac{\partial^2 f}{\partial K^2}(K, T)dT(F_t - K)dK\mu(dx, dt). \tag{12}$$

Setting $\int_t^\tau \frac{\partial^2 f}{\partial K^2}(K, T)dT = \ell(K, t)$ and differentiating w.r.t. t implies:

$$\frac{\partial^2 f}{\partial K^2}(K, t) = -\frac{\partial \ell}{\partial t}(K, t), \quad K > 0, t \in [0, \tau]. \tag{13}$$

Thus, the number of out-of-the-money options to hold at each strike K and maturity $t \in [0, \tau]$ is given by the negative of the time derivative of the desired local liquidity rate at price K and time t. Intuitively, since each option of maturity $T > t$ continues to provide liquidity at its strike for all times $u \in (t, T]$, it is only the options of maturity t and strike K which can cause the local liquidity function to change over time at the price K. As time rolls forward, the absence of a positive position in these options will lead to a decline in the local liquidity. As a special case, when the desired local liquidity function is of the form $\ell(K, t) = f''(K)1(t < T)$ for $t \in [0, \tau]$, then we recover the result that the number of options to hold is $f''(K)dK$ for options of maturity T and zero otherwise.

III-B Local Liquidity Surfaces

The last section showed that in return for initially paying the time value of an option of maturity T and strike K, an investor can lock in the payoffs arising from enjoying local liquidity in one forward contract at the strike K and over the option's life $[0, T]$. Differentiating with respect to T implies that by paying the maturity derivative of the option's time value initially, an investor inherits local liquidity in one forward contract at the strike K and at the future time T. If options of all strikes $K > 0$ and maturities $T \in [0, \tau]$ are available for investment, then it follows that one can synthesize any desired local liquidity function $\ell(F, t), F > 0, t \in [0, \tau]$, as shown in the last subsection.

To express this fundamental result more formally, assume zero interest rates in this subsection for simplicity. Differentiating (1) w.r.t. T implies that the P&L from this spreading strategy is given by:

$$\begin{aligned} \tilde{P\&L}_T &= -\frac{\partial C_0(K,T)}{\partial T} + \frac{1}{2}\frac{\partial L_T^F(K)}{\partial T} \\ &\quad + \left[1(F_{T-} \leq K)(F_T - K)^+ + 1(F_{T-} > K)(K - F_T)^+\right], \end{aligned} \tag{14}$$

where $\tilde{P\&L}_T \equiv \frac{\partial P\&L_T}{\partial T}$. By put call parity, $\frac{\partial C_0(K,T)}{\partial T} = \frac{\partial P_0(K,T)}{\partial T}$, so one may restrict attention to only out-of-the-money options if desired. Consistent with time decay defined as $\frac{\partial C_t}{\partial t}$, we term $\frac{\partial C_0(K,T)}{\partial T}$ the *maturity gain*. From (14), the maturity gain is the risk-neutral expected value of two components. The first component is $\frac{1}{2}\frac{\partial L_T^F(K)}{\partial T}$ whose risk-neutral expected value is closely related to the local (normal) variance rate. The second component is $[1(F_{T-} \leq K)(F_T - K)^+ + 1(F_{T-} > K)(K - F_T)^+]$ whose risk-neutral expected value is closely related to the double integral of the risk-neutral Lévy density governing price jumps, as will be explored in future work.

In principle, the initial price for locking in local liquidity is observable at time 0 for every strike $K > 0$ and maturity $T \in [0, T]$. This initial local liquidity surface would typically resemble a double exponential density with the peak occurring at the current (forward) price and with little variation in T. Mimmicking the analysis in Dupire[4] and others, one can in principle impose a stochastic process on one or more parameters describing this surface, thereby imposing some stochastic evolution for the local liquidity surface. Of course any such process must be consistent with zero risk-neutral drift in $\frac{\partial C_t(K,T)}{\partial T}$.

III-C Achieving a Desired Continuous Payoff

We have seen that the continuously paid payoffs from a breakout option and a price variance swap can both be replicated by combining options of all strikes with a limit order strategy in the underlying. Allowing static positions in options of multiple maturities increases the set of continuously paid payoffs which can be perfectly replicated. This subsection explores the nature of the set of continuously paid payoffs which can be replicated by combining options of all strikes and maturities with a limit order strategy in the underlying.

Suppose an investor wishes to have the term:

$$\int_0^\tau \int_{-\infty}^{\infty} g(F_t; F_{t-}, t)\mu(dx, dt) \tag{15}$$

paid to him at some fixed time $\tau > 0$, where the function $g(F; F_-, t)$ maps the post jump price F, the pre-jump price F_-, and the time t into the continuous payoff realized at t and paid at τ. Since the investor can achieve any desired local liquidity rate $\ell(K, t)$, (11) and (15) imply that the problem is to choose this local liquidity function so that:

$$\int_{F_-}^{F} \ell(K, t)(F - K)dK = g(F; F_-, t), \quad t \in [0, \tau], F > 0, F_- > 0. \tag{16}$$

Theorem 3: *Assume that we have frictionless markets in the asset underlying a continuum of options of strikes $K > 0$ and maturities $T \in [0, \tau]$. Also assume a semi-martingale process for the forward price. If (16) holds for some local liquidity function $\ell(K, t)$, then the continuous payout function $g(F; F_-, t)$ must satisfy:*

$$g(F_-; F_-, t) = 0 \quad t \in [0, \tau], F_- > 0, \tag{17}$$

and:

$$g_1(F_-; F_-, t) = 0, \quad t \in [0, \tau], F_- > 0. \tag{18}$$

Furthermore, $g_{11}(F; F_-, t)$ must be independent of F_-. If g satisfies these 3 properties, then the solution to (16) is:

$$\ell(F, t) = g_{11}(F; F_-, t), \quad t \in [0, \tau], F > 0, F_- > 0. \tag{19}$$

Proof: Setting $F = F_-$ in (16) implies that g vanishes along this diagonal as indicated in (17). Differentiating (16) w.r.t. F implies:

$$\int_{F_-}^{F} \ell(K, t) dK = g_1(F; F_-, t), \quad t \in [0, \tau], F > 0, F_- > 0. \tag{20}$$

Setting $F = F_-$ implies that g_1 also vanishes along this diagonal as indicated in (18). Differentiating (20) w.r.t. F implies (19). Since ℓ does not depend on F_-, a necessary condition on the form of g is that $g_{11}(F; F_-, t)$ is independent of F_-. If $g(F; F_-, t)$ satisfies this condition along with (17) to (18), then (19) implies that the local liquidity rate should be chosen to be the second derivative of the cash flow w.r.t. the post jump price. **QED**

From (13), the number of out-of-the-money options to hold of maturity t and strike K is therefore given by:

$$\frac{\partial^2 f}{\partial K^2}(K, t) = -\frac{\partial g_{11}}{\partial t}(K, t) = -\frac{\partial^3 g}{\partial t \partial K^2}(K, t), \quad K > 0, t \in [0, \tau]. \tag{21}$$

Integrating (21) twice w.r.t. K implies that when considered as function of the spatial variable, the final payoff f and the continuous payoff g differ by at most a linear function of K. One can easily verify that the hedges for the breakout option and the price variance swap are consistent with this general result.

On the negative side, (19) implies that one cannot use *static* option positions to generate cash flows g with g_{11} dependent on F_-. For example, the contracts with continuous payout $\left[\ln\left(\frac{F_t}{F_{t-}}\right)\right]^2$ or $\left[\frac{F_t-F_{t-}}{F_{t-}}\right]^2$ or $[F_t - F_{t-}]^3$ cannot be replicated in this manner. Similarly, a function of the increment, $h(F - F_-, t) \equiv g(F, F_-, t)$, cannot be a polynomial of order greater than two. On the positive side, so long as g satisfies (17) and (18), then *dynamic* trading in options delivers this cash flow, where the position in the option of strike K and maturity t is now:

$$\ell(F, F_-t) = g_{11}(F; F_-, t), \quad t \in [0, \tau], F > 0, F_- > 0. \tag{22}$$

For example, the contracts with continuous payout $\left[\ln\left(\frac{F_t}{F_{t-}}\right)\right]^2$ or $\left[\frac{F_t-F_{t-}}{F_{t-}}\right]^2$ or $[F_t - F_{t-}]^3$ can be replicated by setting the exposure to $2\frac{1-\ln(F_t/F_{t-})}{F_t^2}$ or $\frac{2}{F_{t-}^2}$ or $6(F_t - F_{t-})$ respectively.

Note that each option position depends only on the price and time and is *independent* of the characteristics of the semi-martingale. In particular, any function h of the increment $F - F_-$ can be perfectly replicated by dynamic trading in this manner. However, the cost of creating the dynamic option position will depend on future option prices and so one cannot value general claims without resorting to some restriction on the forward price process, which in turn restricts the range of future option prices.

IV Lévy Hedging

The assumption of a continuum of strikes and maturities may be quite unrealistic in certain option markets. For this reason, we return to an options market where the initial liquidity is only in a single call with fixed strike K and fixed maturity T. In this section, we present a second method called Lévy Hedging for extending the local liquidity arising from a static position in this call to a continuum of price levels. In order to include the standard Black model analysis as a special case of our results, we now assume that $X_t \equiv \ln F_t$ is a semi-martingale under the statistical probability measure P. This formulation has the drawback of preventing bankrupty, but the virtue of comprehending the market's intense focus on implied volatilities, as we will show. If the underlying is a stock index, a currency, or a commodity, then our restriction to positive prices is economically attractive.

From Shiryaev[8], pg. 670, we may associate with the process X a triplet $\mathcal{T} \equiv (B, Q, \nu)$ of *predictable characteristics*, where $B_t \equiv \int_0^t b_s ds$ is the process accumulating drift in the log, $Q_t \equiv \int_0^t \sigma_s^2 ds$ is the quadratic variation in the log, and $\nu[(0,t] \times dx; \omega]$ is the *predictable jump compensator*. For each $\theta \in \Re$ and truncation function $g(x)$, it is useful to associate with $\mathcal{T}$ the following *cumulant* of X:

$$\Psi(\theta)_t = i\theta B_t - \frac{\theta^2}{2} Q_t + \int_{-\infty}^{\infty} [e^{i\theta x} - 1 - i\theta g(x)] \nu[(0,t] \times dx; \omega],$$

which is a predictable process of bounded variation. A truncation function is a function which behaves like x near $x = 0$ and the choice of the truncation function $g(x)$ affects only the process B; the processes Q and ν are independent of the choice of $g(x)$. For simplicity, we assume that the process X is quasi left continuous, so that the cumulant process $\Psi(\theta)_t$ is continuous in t. The term cumulant then arises because the process $\frac{e^{i\theta X_t}}{e^{\Psi(\theta)_t}}$ is a local martingale.

Since the call is assumed to not trade between time 0 and T, the option buyer uses *value functions* to determine intermediate P&L and to hedge. For simplicity, we restrict attention to marking and hedging value functions $C(F, t)$ which are Markov in the forward price F and time t. In order to retain unrestricted interest rates, we work with *forward value functions*, which obviously indicate the forward value of the call.

We may thus distinguish between two types of forward value functions:

1. The pricing valuation function $C^p(F, t) \equiv E^p[S_T - K]^+ | F_t = F\}$, based on a pricing measure Q^p, which the investor uses to mark the option position.

2. The hedging valuation function $C^h(F, t) \equiv E^h[S_T - K]^+ | F_t = F\}$, based on a hedging measure Q^h, which the investor uses to hedge the option position.

We assume that the market spot option price C_0 is arbitrage-free and that the *initial* forward pricing function matches the market forward price of the call, i.e. $C^p(F_0, 0) = \frac{C_0}{B_0}$. For tractability reasons, we assume that X is a Lévy process under both the pricing measure Q^p and the hedging measure Q^h. More specifically, we assume that under the pricing measure Q^p, the characteristic function of X_t is:

$$E^p e^{i\theta X_t} = e^{t\Psi_p(\theta)}, \tag{23}$$

where by the Lévy-Khintchine formula, the characteristic exponent is:

$$\Psi_p(\theta) = i\theta b_p - \frac{\theta^2}{2}\sigma_p^2 + \int_{-\infty}^{\infty} [e^{i\theta x} - 1 - i\theta g_p(x)]\nu_p(dx). \tag{24}$$

Here, $\sigma_p > 0$ is the option buyer's constant pricing volatility, and $\nu_p(\cdot) > 0$ is the buyer's constant pricing Lévy measure. The constant drift in the log b_p is determined by σ_p and ν_p to be:

$$b_p = -\frac{\sigma_p^2}{2} - \int_{-\infty}^{\infty} [e^x - 1 - g_p(x)]\nu_p(dx), \tag{25}$$

as proved in Appendix 1. The triplet $(b_p, \sigma_p, \nu_p(dx))$ is referred to jointly as the option buyer's *pricing characteristics* of the Lévy process.

Under the hedging measure Q^h, the characteristic function of X_t is thought to be:

$$E^h e^{i\theta X_t} = e^{t\Psi_h(\theta)},$$

where the characteristic exponent is:

$$\Psi_h(\theta) = i\theta b_h - \frac{\theta^2}{2}\sigma_h^2 + \int_{-\infty}^{\infty} [e^{i\theta x} - 1 - i\theta g_h(x)]\nu_h(dx).$$

Here, $\sigma_h > 0$ is the option buyer's constant hedge volatility, and $\nu_v(\cdot) > 0$ is the buyer's constant hedge Lévy measure. The constant drift in the log b_h is similarly determined by σ_h and ν_h to be:

$$b_h = -\frac{\sigma_h^2}{2} - \int_{-\infty}^{\infty} [e^x - 1 - g_h(x)]\nu_h(dx). \tag{26}$$

The triplet $(b_h, \sigma_h, \nu_h(dx))$ is referred to jointly as the investor's *hedge characteristics* of the Lévy process.

Given the pricing function $C^p(F, t) \equiv E^p\{[S_T - K]^+ | F_t = F\}$, we assume that the call with time T payoff $[S_T - K]^+$ is bought at time $t = 0$ for $C^p(F_0, 0)$ and that the number of forwards held short over $[0, T)$ is the first partial derivative of C^h w.r.t. F. Appendix 2 proves a generalization of the following theorem:

Theorem 4: *Assuming frictionless markets in a call's underlying asset and a semi-martingale process for the forward price, consider the P&L from buying a static position in a call at time* 0 *which pays* *off* $[S_T - K]^+$ *at* T. *Suppose that the call is purchased for* C_0 *initially and that the number of forward contracts held over* $[0,T)$ *is the negative of the first partial derivative of* C^h *w.r.t.* F, *where* $C^h(F,t) \equiv E^h[S_T - K]^+|F_t = F\}$. *Then the terminal P&L is:*

$$\begin{aligned} P\&L_T &= -[C^p(F_0,0) - C^h(F_0,0)] + \int_0^T (\sigma_{t-}^2 - \sigma_h^2)\frac{F_{t-}^2}{2}\frac{\partial^2 C^h}{\partial F^2}(F_{t-},t)dt \\ &+ \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 C^h}{\partial F^2}(L,t)(F_{t-}e^x - L)dL[\mu(dx,dt) - \nu^h(dx)dt]. \end{aligned} \tag{27}$$

We again focus on interpreting the innermost integral appearing in the last term in (27). The quantity traded at each price level L between F_{t-} to $F_{t-}e^x$ is now $\frac{\partial^2 C^h}{\partial F^2}(L,t)dL$, which is positive for an option as proved in Appendix 3. Furthermore, we now show that the integral $\int_0^\infty \frac{\partial^2 C^h}{\partial F^2}(L,t)dL$ is one, so that $\frac{\partial^2 C^h}{\partial F^2}(L,t)$ can properly be termed a local liquidity *density*. To prove the result, note that $\frac{\partial^2 C^h}{\partial F^2}$ is the derivative of $\frac{\partial C^h}{\partial F}$, which by Appendix 3 has the simple representation:

$$\frac{\partial C^h}{\partial F}(L,t) = Q^{(1)}(S_T > K|F_t = L),$$

where under the probability measure $Q^{(1)}$, the characteristics of $\ln F$ are now $(\frac{\sigma_h^2}{2} - \int_{-\infty}^{\infty}(e^x - xe^x - 1)\nu^h(dx), \sigma_h, e^x\nu_h(dx))$. The above implies that:

$$\begin{aligned} \int_0^\infty \frac{\partial^2 C^h}{\partial F^2}(L,t)dL &= \int_0^\infty \frac{\partial}{\partial L}Q^{(1)}(S_T > K|F_t = L)dL \\ &= Q^{(1)}(S_T > K|F_t = \infty) - Q^{(1)}(S_T > K|F_t = 0) \\ &= 1 - 0 = 1. \end{aligned}$$

Thus, the final P&L can be re-written in terms of the local liquidity density $p_t(L) \equiv \frac{\partial^2 C^h}{\partial F^2}(L,t)$ as:

$$\begin{aligned} P\&L_T &= -[C^h(F_0,0) - C^p(F_0,0)] + \int_0^T (\sigma_{t-}^2 - \sigma_h^2)\frac{F_{t-}^2}{2}p_t(F_{t-})dt \\ &+ \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} p_t(L)(F_{t-}e^x - L)dL[\mu(dx,dt) - \nu^h(dx)dt]. \end{aligned} \tag{28}$$

where for each t, $p_t(L)$ is a proper probability density function (PDF). Put call parity implies that puts and calls have the same gamma, and so the liquidity density concept applies equally well to puts.

We can again recover the results of Section I as the special case when the hedge characteristics degenerate to $(b_h, \sigma_h, \nu_h(dx)) = (0, 0, 0)$. In this case, when given that the forward price at t is F, the forward price process under the hedging measure is the deterministic process $F_u = F, u \geq t$. The forward value function under the hedging measure reduces to:

$$C^h(F, t) = \max[0, F - K], \tag{29}$$

which is independent of t. Differentiating (29) twice w.r.t. F implies:

$$\frac{d^2 C^h}{dF^2}(F, t) = \delta(F - K), \tag{30}$$

where $\delta(\cdot)$ is once again the Dirac delta function. Substituting (29) and (30) into Theorem 4 recovers the result of Theorem 1:

$$\begin{aligned} P\&L_T &= -[C_0 e^{rT} - (F_{0T} - K)^+] + \frac{1}{2} L_T^F(K) \\ &+ \sum_{0<t\leq T} \left[1(F_{t-} \leq K)(F_t - K)^+ + 1(F_{t-} > K)(K - F_t)^+\right]. \end{aligned} \tag{31}$$

Thus, by using a (non-degenerate) Lévy hedging strategy an investor spreads out the one forward contract traded at the option's strike into a proper density with support on all positive levels. The initial investment in the strategy will be lower, but this reduction in initial cost is paid for by continuous payments through time. The magnitude of these costs depends on the price level and on the hedging volatility and Lévy density, but not on the realizations of the stochastic volatility or jump arrival rate process.

V Comparing and Combining Approaches

V-A Comparing Approaches

Hedging the intrinsic value of a continuum of strikes and Lévy hedging a single call both accomplish the objective of spreading out the local liquidity to a continuum of prices. It is instructive to compare the P&L from hedging the intrinsic value of a continuum of strikes:

$$P\&L_T = -\left[\frac{V_0^f}{B_0} - f(F_0)\right] + \frac{1}{2}\int_0^\infty L_T^F(K) m(dK) + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_t} f''(K)(F_t - K) dK \mu(dx, dt), \tag{32}$$

with the P&L from holding just one call, and Lévy hedging with the hedging measure chosen so that $C^h(F_0,0) = C^p(F_0,0)$:

$$
\begin{aligned}
P\&L_T &= \int_0^T (\sigma_{t-}^2 - \sigma_h^2)\frac{F_{t-}^2}{2}\frac{\partial^2 C^h}{\partial F^2}(F_{t-},t)dt \\
&+ \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 C^h}{\partial F^2}(L,t)(F_{t-}e^x - L)dL[\mu(dx,dt) - \nu^h(dx)dt].
\end{aligned} \tag{33}
$$

When f is convex, the first approach has a strictly positive initial cost[10] and generates strictly nonnegative cash flows. In contrast, the second approach has zero initial cost and generates cash flows with positive probability of both losses and gains at each t. In the second approach, the benefits enjoyed by having liquidity and optionality based on the actual stochastic variance rate σ_t^2 and arrival rate $\mu(dx,dt)$ are financed by selling these benefits based on some pairing of a constant variance rate σ_h^2 and Lévy density $\nu^h(dx)dt$ consistent with the initial price. Thus, the net cash flow results from the difference between the stochastic variance and arrival rates and the time decay as measured in terms of constant volatility and arrival rate. The delta-hedged option is therefore a vehicle for trading volatility and/or arrival rates where at each t, the size of each bet depends on the pre and post jump prices. If one can suppress this dependence on price, then one can synthesize a pure variance or Lévy measure swap. However, when Lévy hedging a single call, the floating and fixed payments must depend on the price, since the volume profile is a probability density, which is never uniform. In contrast, through various choices of f, the other approach can generate nonnegative cashflows through time which can assume a wide variety of profiles. This suggests that interesting products can be created by combining the two approaches which we next explore.

V-B Combining Approaches

This subsection explores the benefits of Lévy hedging a continuum of options. Thus, we now assume that there exists initial liquidity in a continuum of strikes of a fixed maturity. We again distinguish between the two types of value functions:

[10]The positivity of initial cost is a consequence of Jensen's inequality.

1. The pricing valuation function $V^p(F,t) \equiv E^p[f(F_T)|F_t = F]$, based on a pricing measure Q^p, which the investor uses to mark the option portfolio.

2. The hedging valuation function $V^h(F,t) \equiv E^h[f(F_T)|F_t = F]$, based on a hedging measure Q^h, which the investor uses to determine hedges for the option portfolio.

Given the forward pricing function $V^p(F,t) \equiv E^p[f(F_T)|F_t = F]$, we assume that a claim with time T payoff $f(S_T)$ is bought at time $t = 0$ for the arbitrage-free price of $V^p(F_0, 0)$ and that delta-hedging is conducted over $[0,T)$ using the first partial derivative of V^h w.r.t. F. Appendix 2 proves the following theorem:

Theorem 5: *Consider the P&L from buying a static position in options at time* 0 *which pays off* $f(S_T)$ *at* T. *Suppose that the options are purchased for forward delivery at* $V^p(F_0,0) \equiv E^p[f(S_T)|F_t = F]$ *initially and that the number of forward contracts held over* $[0,T)$ *is the negative of the first partial derivative of* V^h *w.r.t.* F*, where* $V^h(F,t) \equiv E^h[f(S_T)|F_t = F]$*. Then the terminal P&L is:*

$$\begin{aligned} P\&L_T &= -[V^p(F_0,0) - V^h(F_0,0)] + \int_0^T (\sigma_{t-}^2 - \sigma_h^2) \frac{F_{t-}^2}{2} \frac{\partial^2 V^h}{\partial F^2}(F_{t-},t)dt \\ &\quad + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 V^h}{\partial F^2}(K,t)(F_{t-}e^x - K)dK[\mu(dx,dt) - \nu^h(dx)dt]. \end{aligned} \tag{34}$$

Thus, the P&L arises from the difference between the amount $V^h(F_0,0)$ received at T from the hedge strategy and the forward value of the premium paid, $V^p(F_0,0)$, and also from the remainder term in the first order Taylor series expansion of $V^h(F,t)$. In principle, one can choose the payoff function $f(F)$, the hedge volatility σ_h and the hedge Lévy measure $\nu^h(dx)$ so that the P&L distribution is optimal for strategies in this class.

V-C Implications for Risk Management

This subsection shows that for any risk-neutral measure, the risk-neutral expected P&L is independent of the choice of hedge characteristics. To demonstrate this, let Q be a risk-neutral measure and define the risk-neutral expected value of a random variable as the initial cost of creating the random variable using a self-financing strategy. Then taking risk-neutral expectations of both sides of (48):

$$E_0^Q P\&L_T = E_0^Q f(F_t) - V^p(F_0, 0),$$

since $E_0^Q dF_t = 0$ for all $t \in [0, T]$. The RHS is independent of the hedge volatility σ_h and the hedge Lévy measure $\nu^h(x)$. It follows that the choice of these hedge characteristics is irrelevant for risk-neutral expected P&L. For each possible risk-neutral measure, the risk-neutral expected P&L is determined at inception and equals the difference between its fair forward price, $E_0^Q f(S_T)$ and the forward value of its cost, $V^p(F_0, 0)$.

For any risk-neutral measure Q^m, let $\nu^m(dx)$ denote the associated Lévy measure. When regarded as a function of T, $P\&L_T$ is a semimartingale under Q^m. We can therefore decompose it into a process of bounded variation and a Q^m martingale:

$$\begin{aligned} P\&L_T \quad = \quad & V^h(F0,0) - V^p(F0,0) + \int_0^T [\sigma_{t-}^2 - \sigma_h^2] \frac{F_{t-}^2}{2} \frac{\partial^2 V^h}{\partial F^2}(F_{t-}, t)dt + \\ & \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 V}{\partial F^2}(L, t-)(F_{t-}e^x - L)dL[\nu^m(dx, dt) - \nu^h(dx)dt] \\ & + M_T^h, \quad T > 0, \end{aligned} \tag{35}$$

where M_T^h is the compensated jump martingale:

$$M_T^h \equiv \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 V}{\partial F^2}(L, t-)(F_{t-}e^x - L)dL[\mu(dx, dt) - \nu^m(dx)dt], T > 0.$$

Appendix 3 proves that if the payoff function $f(F)$ is convex, then the forward value function $V^h(F, t)$ is also convex in F. Thus, if the volatility rate σ_t is bounded below and if one can hedge with $\sigma_h < \sigma_t$ and $\nu^h(dx)dt < \nu(dx, dt)$, and if one can still buy the claim for $V^p(F_0, 0) < V^h(F_0, 0)$, then the risk-neutral expected P&L is positive. If one is assured that $\nu(dx) < \mu(dx)$, then the realized P&L will be positive.

For a general payoff function, if σ_t^2 is constant at σ^2 and we set $\sigma_h^2 = \sigma^2$ and $\nu_h(dx) = \nu(dx)$, then the P&L is simply:

$$P\&L_T = V^h(F_0, 0) - V^p(F_0, 0) + M_T^h.$$

Thus, the only way for the P&L to be riskless is if $\mu(dx) = 0$ (no jumps) and $\sigma_t = \sigma$ (constant vol) and in this case, we would set $\nu_h(dx) = 0$ and $\sigma_h = \sigma$.

V-D Black Hedging

We define Black hedging as the special case of Lévy hedging when $\nu^h(dx) = 0$, i.e. when the investor uses the Black model to hedge. This subsection shows that there exists a portfolio of options of all strikes and maturities, which when Black hedged, results in an arbitrary local liquidity density.

From Theorem 5, the terminal P&L from Black hedging is:

$$\begin{aligned} P\&L_T &= -[V^p(F_0,0) - V^h(F_0,0)] + \int_0^T (\sigma_{t-}^2 - \sigma_h^2)\frac{F_{t-}^2}{2}\frac{\partial^2 V^h}{\partial F^2}(F_{t-},t)dt \\ &+ \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 V^h}{\partial F^2}(F,t)(F_{t-}e^x - F)dF\mu(dx,dt), \end{aligned} \tag{36}$$

where under the measure Q^h:

$$\frac{dF_t}{F_t} = \sigma_h dW_t, \quad t \in [0,T].$$

Put call parity and (3) imply that the gamma appearing in the last term can be written as:

$$\frac{\partial^2 V^h}{\partial F^2}(F,t) = \int_0^\infty \frac{\partial^2 f(K,T)}{\partial K^2}\frac{\partial^2 C(t,F;K,T)}{\partial F^2}dK, \tag{37}$$

where $C(t,F;K,T)$ is the Black model value at t of a call with strike K and maturity T when $F_t = F$. Since the call gamma $\frac{\partial^2 C(t,F;K,T)}{\partial F^2}$ is a lognormal density in K, the change of variable $k = \ln K$ transforms it into a normal density in $x = \ln F$ with mean $k - \frac{\sigma_h^2(T-t)}{2}$ and variance $\frac{\sigma_h^2(T-t)}{2}$. By a judicious choice[11] of $f(K,T)$, the gamma of the T maturity option portfolio can be rewritten as:

$$\frac{\partial^2 V^h}{\partial F^2} = \int_{-\infty}^{\infty} p(k) n_x(k, \sigma_h^2(T-t)dk, \tag{38}$$

where $p(k)$ is a PDF in k and $n_x(k,v)$ is a normal density in x with mean k and variance v. Multiplying (36) by a PDF $\tilde{p}$ in T, integrating T from 0 to ∞, and employing Fubini's theorem implies that the gamma

[11]One chooses f so that $\frac{\partial^2 f(K,T)}{\partial K^2}$ both eliminates the Jacobian and changes the mean of the normal to k.

of the portfolio of all strikes and maturities is now:

$$\int_t^\infty \tilde{p}(T) \int_{-\infty}^{\infty} p(k) n_x(k, \sigma_h^2(T-t) dk dT$$

A final change of variables $v = \sigma_h^2(T-t)$ results in a mixture of normals, where the mixing is done separately over both mean and variance. Wiener showed that this mixture can be used to generate any PDF.

Note that this result differs slightly from the one in subsection III-A showing that any local liquidity function can be created by Black hedging a continuum of strikes and maturities at zero volatility. The content of the above result is that if the target local liquidity function is any given PDF, then the number of strikes held and the number of maturities held to create this PDF via Black hedging are each representable as L^1 positive functions defined on $\Re^+$.

V-E Return Variance Swap

A return variance swap is an example of a generalized swap where the dependence on the stock price is suppressed. Perfect replication requires that $\mu = 0$ (no jumps) and the hedge sets $\nu^h(dx) = 0$ accordingly. By choosing the payoff to be twice the difference between the discretely compounded return and the continuously compounded return:

$$f(F) = 2\left[\frac{F}{F_0} - 1 - \ln\left(\frac{F}{F_0}\right)\right],$$

the forward valuation function simplifies to:

$$V^h(F,t) = 2\left[\frac{F}{F_0} - 1 - \ln\left(\frac{F}{F_0}\right)\right] + \sigma_h^2(T-t),$$

where σ_h can be chosen so that $V^h(F_0, 0) = V^p(F_0, 0)$. Since $\frac{\partial^2 V^h}{\partial F^2}(F,t) = \frac{2}{F^2}$, the P&L in Theorem 5 reduces to the payoff from a (continuously monitored) return variance swap:

$$P\&L_T = \int_0^T (\sigma_{t-}^2 - \sigma_h^2) dt. \tag{39}$$

Since the payoff can also be represented as:

$$f(F) = \int_0^{F_0-} \frac{2}{K^2}(K-F)^+ dK + \int_{F_0+}^{\infty} \frac{2}{K^2}(F-K)^+ dK,$$

the static component of the hedge holds $\frac{2}{K^2}dK$ of each of the initially out-of-the-money options. Since $\frac{\partial V^h}{\partial F}(F,t) = 2\left[\frac{1}{F_0} - \frac{1}{F_t}\right]$, the negative of this quantity represents the number of forwards held. When jumps are possible, Theorem 5 implies that the P&L from delta-hedging the option portfolio is:

$$P\&L_T = \int_0^T (\sigma_{t-}^2 - \sigma_h^2)dt + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{2}{L^2}(F_{t-}e^x - L)dL\mu(dx,dt). \tag{40}$$

The last integral simplifies to $\int_0^T \int_{-\infty}^{\infty} 2(e^x - 1 - x)\mu(dx,dt)$, so that if the investor is also short a variance swap paying $\int_0^T \left[\int_{-\infty}^{\infty} x^2\mu(dx,dt) + \sigma_{t-}^2 - \sigma_h^2\right] dt$ at T, the total hedging error due to jumps is $\int_0^T \int_{-\infty}^{\infty} 2\left[e^x - 1 - x - \frac{x^2}{2}\right]\mu(dx,dt)$. The hedging error at each time t can be partially eliminated by projecting it on the payoff of 1 from an annuity, $e^x - 1$ from $\frac{1}{F_{t-}}$ futures, and the payoff x^2 from the variance swap. To leading order, the hedging error at each time t is $\frac{1}{3}x^3$, suggesting that a skewness swap payoff can be approximated when there are jumps by shorting 3 variance swaps, and hedging the continuous movements as described above.

Carr and Madan[2] also consider corridor variance swaps which pay the realized variance while the underlying is between two levels and between two dates. By shrinking the width of the corridor and the term of the contract, they also show how to create local variance swaps, which form the basis for the unified theory of volatility expounded in Dupire[4].

VI Summary and Extensions

In this paper, we characterized an option's time value as the (risk-neutral) expected benefit from being able to buy or sell the underlying at each cross of the option's strike. Thus, an option provides liquidity at its strike even when the market doesn't. This liquidity is combined with the ability to base the buy/sell decision on the post jump price, so that rational investors buy on rises and sell on drops. We then presented two methods for extending this liquidity to every price between the pre and post jump level. The first method involved holding a continuum of options of all strikes and delta hedging as if the price process were deterministic. The second method holds one option, but uses a Lévy hedging strategy. We discussed the advantages and disadvantages of each approach and considered the benefits of combining them. An analysis of the P&L generated by the various strategies suggested three new financial products which can be robustly hedged. In particular, the breakout option and the price variance swap are robust to stochastic volatility and arrival rates, while a (corridored) return variance swap requires the absence of jumps.

A fairly straightforward extension to this work would involve similar analyses involving static positions in American or exotic options. A more challenging extension to this work would involve investigating the nature of payoffs which can be robustly hedged via dynamic strategies in options. In the interests of brevity, these extensions are left for future research.

Appendix 1: Determining the Risk-Neutral Drift in the Log

Recall from (23) and (24) that for a Lévy process X, the characteristic function of the random variable X_t under the investor's pricing measure Q^p is:

$$\phi(\theta) \equiv E^p e^{i\theta X_t} = e^{t\Psi_p(\theta)},$$

where by the Lévy-Khintchine formula, the characteristic exponent is:

$$\Psi_p(\theta) = i\theta b_p - \frac{\theta^2}{2}\sigma_p^2 + \int_{-\infty}^{\infty} [e^{i\theta x} - 1 - i\theta g_p(x)]\nu_p(dx).$$

Evaluating $\Psi(\cdot)$ at $\theta = \frac{1}{i}$ implies:

$$E^p e^{X_t} = e^{t\left\{b_p + \frac{\sigma_p^2}{2} + \int_{-\infty}^{\infty} [e^x - 1 - g_p(x)]\nu_p(dx)\right\}}.$$

We wish to choose b_p so that the process $e^{X_t}, t > 0$ is a Q^p martingale started at one. Setting $E^p e^{X_t} = 1$ implies:

$$b_p = -\frac{\sigma_p^2}{2} - \int_{-\infty}^{\infty} [e^x - 1 - g_p(x)]\nu_p(dx).$$

Appendix 2: Proof of Theorem 5

Using Itô's lemma for semi-martingales (see Shiryaev[8] pg. 307) on $V^h(F,t)$:

$$\begin{aligned} V^h(F_T,T) &= V^h(F_0,0) + \int_0^T \frac{\partial V^h}{\partial F}(F_{t-},t)dF_t \\ &+ \int_0^T \int_{-\infty}^{\infty} \left[V^h(F_{t-}e^x,t) - V^h(F_{t-},t) - \frac{\partial V^h}{\partial F}(F_{t-},t)F_{t-}(e^x-1) \right] \mu(dx,dt) \\ &+ \int_0^T \left[\frac{\partial V^h}{\partial t}(F_{t-},t) + \frac{\sigma_{t-}^2 F_{t-}^2}{2} \frac{\partial^2 V^h}{\partial F^2}(F_{t-},t) \right] dt, \end{aligned} \tag{41}$$

where $\mu(dx,dt)$ is the integer valued random measure which counts the number of jumps in any region of space-time.

We now prove that there are five equivalent representations for the expression in square brackets in the middle line of (41):

$$\begin{aligned} & V^h(F_{t-}e^x,t) - V^h(F_{t-},t) - \frac{\partial V^h}{\partial F}(F_{t-},t)F_{t-}(e^x-1) \\ = & \sum_{j=2}^{\infty} \frac{\partial^j V^h(F_{t-},t)}{\partial F^j}[F_{t-}(e^x-1)]^j \\ = & \frac{\partial^2 V^h}{\partial F^2}(F_t^*,t)F_{t-}^2(e^x-1)^2 \\ = & \int_{F_{t-}}^{\infty} \frac{\partial^2 V^h}{\partial F^2}(K,t)(F_{t-}e^x-K)^+dK + \int_0^{F_{t-}} \frac{\partial^2 V^h}{\partial F^2}(K,t)(K-F_{t-}e^x)^+dK \\ = & \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 V^h}{\partial F^2}(K,t)(F_{t-}e^x-K)dK \\ = & \int_{F_{t-}}^{F_{t-}e^x} \int_{F_{t-}}^{K} \frac{\partial^2 V^h}{\partial F^2}(L,t)dLdK. \end{aligned} \tag{42}$$

The first equality follows from the analytic nature of V and from Taylor series. The second equality follows from the standard form of Taylor series with remainder, where F_t^* is known to be between F_{t-} and $F_{t-}e^x$. The third equality follows from a less standard form of Taylor series with remainder which is proved in Appendix 4. The final equality follows from differentiating this remainder term w.r.t. F_{t-} and integrating back, where the constant of integration is determined from requiring that the expression vanish for $x = 0$.

The last line can be shown to be equal to the second last line by integrating the outer integral by parts. Substituting the second last line of (42) into the square brackets in the middle line of (41) leaves:

$$\begin{aligned} V^h(F_T,T) &= V^h(F_0,0)+\int_0^T \frac{\partial V^h}{\partial F}(F_{t-},t)dF_t \\ &+\int_0^T\int_{-\infty}^{\infty}\int_{F_{t-}}^{F_{t-}e^x}\frac{\partial^2 V}{\partial F^2}(K,t)(F_{t-}e^x-K)dK\mu(dx,dt) \\ &+\int_0^T\left[\frac{\partial V^h}{\partial t}(F_{t-},t)+\frac{\sigma_{t-}^2F_{t-}^2}{2}\frac{\partial^2 V^h}{\partial F^2}(F_{t-},t)\right]dt, \end{aligned} \tag{43}$$

Since the process $V^h(F_t,t)$ is a martingale under Q^h, the value function $V^h(F,t)$ satisfies the following partial integro-differential equation:

$$\begin{aligned} &\int_{-\infty}^{\infty}\left[V^h(Fe^x,t)-V^h(F,t)-\frac{\partial V^h}{\partial F}(F,t)F(e^x-1)\right]\nu^h(dx) \\ &+\frac{\partial V^h}{\partial t}(F,t)+\frac{\sigma_h^2F^2}{2}\frac{\partial^2 V^h}{\partial F^2}(F,t)=0. \end{aligned} \tag{44}$$

or equivalently:

$$\int_{-\infty}^{\infty}\int_{F}^{Fe^x}\frac{\partial^2 V^h}{\partial F^2}(K,t)(Fe^x-K)dK\nu^h(dx)+\frac{\partial V^h}{\partial t}(F,t)+\frac{\sigma_h^2F^2}{2}\frac{\partial^2 V^h}{\partial F^2}(F,t)=0. \tag{45}$$

Furthermore, by the definition of V^h, we have:

$$V^h(F,T)=f(F). \tag{46}$$

Substitution of (45) and (46) in (44) implies:

$$\begin{aligned} f(S_T) &= V^h(F_0,0)e^{rT}+\int_0^T \frac{\partial V^h}{\partial F}(F_{t-},t)dF_t \\ &+\int_0^T e^{r(T-t)}(\sigma_{t-}^2-\sigma_h^2)\frac{F_{t-}^2}{2}\frac{\partial^2 V^h}{\partial F^2}(F_{t-},t)dt \\ &+\int_0^T\int_{-\infty}^{\infty}\int_{F_{t-}}^{F_{t-}e^x}\frac{\partial^2 V^h}{\partial F^2}(K,t)(F_{t-}e^x-K)dK[\mu(dx,dt)-\nu^h(dx)dt]. \end{aligned} \tag{47}$$

Define the terminal P&L of the strategy as:

$$P\&L_T \equiv f(S_T)-V^p(F_0,0)-\int_0^T \frac{\partial V^h}{\partial F}(F_{t-},t)dF_t. \tag{48}$$

Substituting (47) in (48) implies:

$$P\&L_T = [V^h(F_0,0) - V^p(F_0,0) + \int_0^T (\sigma_{t-}^2 - \sigma_h^2)\frac{F_{t-}^2}{2}\frac{\partial^2 V^h}{\partial F^2}(F_{t-},t)dt + \int_0^T \int_{-\infty}^{\infty} \int_{F_{t-}}^{F_{t-}e^x} \frac{\partial^2 V^h}{\partial F^2}(K,t)(F_{t-}e^x - K)dK[\mu(dx,dt) - \nu^h(dx)dt]. \tag{49}$$

Appendix 3: Convexity of Value Function

Letting $V_f^h(F,t) \equiv \frac{\partial V^h}{\partial F}(F,t)$, differentiating the partial integro-differential equation (44) w.r.t. F yields:

$$\int_{-\infty}^{\infty}\left[e^x V_f^h(Fe^x,t) - V_f^h(F,t) - \frac{\partial V_f^h}{\partial F}(F,t)F(e^x-1) - V_f^h(F,t)(e^x-1)\right]\nu^h(dx)$$

$$+\frac{\partial V_f^h}{\partial t}(F,t) + \frac{\sigma_h^2 f^2}{2}\frac{\partial^2 V_f^h}{\partial F^2}(F,t) + \sigma_h^2 F\frac{\partial V_f^h}{\partial F}(F,t) = 0. \quad (50)$$

Subtracting and adding $\int_{-\infty}^{\infty} e^x\frac{\partial V_f^h}{\partial F}(F,t)F(e^x-1)\nu^h(dx)$ leaves the partial integro-differential equation:

$$\int_{-\infty}^{\infty}\left[V_f^h(Fe^x,t) - V_f^h(F,t) - \frac{\partial V_f^h}{\partial F}(F,t)F(e^x-1)\right]e^x\nu^h(dx) \quad (51)$$

$$+\frac{\partial V_f^h}{\partial t}(F,t) + \frac{\sigma_h^2 F^2}{2}\frac{\partial^2 V_f^h}{\partial F^2}(F,t) + \left[\sigma_h^2 + \int_{-\infty}^{\infty}(e^x-1)^2\nu^h(dx)\right]F\frac{\partial V_f^h}{\partial F}(F,t) = 0,$$

subject to:

$$V_f^h(F,T) = f'(F). \quad (52)$$

Let $X = \ln F$ be a change of independent variable and let $U_f^h(X,t) \equiv V_f^h(F,t)$ be the function relating the delta to X. Then the partial integro-differential equation (51) transforms to:

$$\int_{-\infty}^{\infty}\left[U_f^h(X+x,t) - U_f^h(X,t) - \frac{\partial U_f^h}{\partial X}(X,t)x\right]e^x\nu^h(dx) \quad (53)$$

$$+\frac{\partial U_f^h}{\partial t}(X,t) + \frac{\sigma_h^2}{2}\frac{\partial^2 U_f^h}{\partial X^2}(X,t) + \left[\frac{\sigma_h^2}{2} - \int_{-\infty}^{\infty}(e^x - xe^x - 1)\nu^h(dx)\right]\frac{\partial U_f^h}{\partial X}(X,t) = 0,$$

subject to:

$$U_f^h(X,T) = f'(e^X). \quad (54)$$

The Feynman Kac theorem can be used to give a probabilistic representation of the solution $U_f^h(X,t)$ using a probability measure $Q^{(1)}$ under which the characteristics of $X = \ln F$ are now $(\frac{\sigma_h^2}{2} - \int_{-\infty}^{\infty}(e^x - xe^x - 1)\nu^h(dx), \sigma_h, e^x\nu_h(dx))$. If the payoff $f(F)$ is increasing, then the value function $V^h(F,t)$ is also increasing in F.

Letting $V^h_{ff}(F,t) \equiv \frac{\partial^2 V^h}{\partial F^2}(F,t)$ denote gamma, differentiating the partial integro-differential equation (52) w.r.t. F and subtracting and adding the term $\int\limits_{-\infty}^{\infty} e^{2x} \frac{\partial V^h_{ff}}{\partial F}(F,t)F(e^x-1)\nu^h(dx)$ leaves:

$$\int\limits_{-\infty}^{\infty}\left[V^h_{ff}(Fe^x,t)-V^h_{ff}(F,t)-\frac{\partial V^h_{ff}}{\partial F}(F,t)F(e^x-1)\right]e^{2x}\nu^h(dx)$$
$$+\frac{\partial V^h_{ff}}{\partial t}(F,t)+\frac{\sigma_h^2 F^2}{2}\frac{\partial^2 V^h_{ff}}{\partial F^2}(F,t)+\left[2\sigma_h^2+\int\limits_{-\infty}^{\infty}(e^x-1)(e^{2x}-1)\nu^h(dx)\right]F\frac{\partial V^h_{ff}}{\partial F}(F,t)$$
$$+\left[\sigma_h^2+\int\limits_{-\infty}^{\infty}(e^x-1)^2\nu^h(dx)\right]V^h_{ff}(F,t)=0, \tag{55}$$

subject to:

$$V^h_{ff}(F,T)=f''(F). \tag{56}$$

Again let $X=\ln F$ be a change of independent variable and let $U^h_{ff}(X,t)\equiv V^h_{ff}(F,t)$ be the function relating the gamma to X. Then the partial integro-differential equation (55) transforms to:

$$\int\limits_{-\infty}^{\infty}\left[U^h_{ff}(X+x,t)-U^h_{ff}(X,t)-\frac{\partial U^h_{ff}}{\partial X}(X,t)x\right]e^{2x}\nu^h(dx) \tag{57}$$
$$+\frac{\partial U^h_{ff}}{\partial t}(X,t)+\frac{\sigma_h^2}{2}\frac{\partial^2 U^h_{ff}}{\partial X^2}(X,t)+\left[\frac{3\sigma_h^2}{2}+\int\limits_{-\infty}^{\infty}(1-e^x+xe^{2x})\nu^h(dx)\right]\frac{\partial U^h_{ff}}{\partial X}(X,t)$$
$$+\left[\sigma_h^2+\int\limits_{-\infty}^{\infty}(e^x-1)^2\nu^h(dx)\right]U^h_{ff}(X,t)=0,$$

subject to:

$$U^h_{ff}(X,T)=f''(e^X). \tag{58}$$

The Feynman Kac theorem can again be used to give a probabilistic representation of the solution $V^h_{ff}(F,t)$ using a probability measure $Q^{(2)}$ under which the characteristics of $\ln F$ are now $(\frac{3\sigma_h^2}{2}+\int\limits_{-\infty}^{\infty}(1-e^x+xe^{2x})\nu^h(dx),\sigma_h,e^{2x}\nu_h(dx))$. If the payoff $f(F)$ is convex, then the value function $V^h(F,t)$ is also convex in F.

Appendix 4: Remainder Term of Taylor Series

The fundamental theorem of calculus implies that for any twice differentiable function f and any fixed κ:

$$\begin{aligned} f(F) &= f(\kappa) + 1_{F>\kappa}\int_\kappa^F f'(L)dL - 1_{F<\kappa}\int_F^\kappa f'(L)dL \\ &= f(\kappa) + 1_{F>\kappa}\int_\kappa^F \left[f'(\kappa) + \int_\kappa^L f''(K)dK\right]dL \\ &\quad -1_{F<\kappa}\int_F^\kappa \left[f'(\kappa) - \int_L^\kappa f''(K)dK\right]dL. \end{aligned}$$

Noting that $f'(\kappa)$ does not depend on L and applying Fubini's theorem:

$$f(F) = f(\kappa) + f'(\kappa)(F-\kappa) + 1_{F>\kappa}\int_\kappa^F\int_K^F f''(K)dLdK + 1_{F<\kappa}\int_F^\kappa\int_F^K f''(K)dLdK.$$

Performing the integral over L yields:

$$\begin{aligned} f(F) &= f(\kappa) + f'(\kappa)(F-\kappa) + 1_{F>\kappa}\int_\kappa^F f''(K)(F-K)dK + 1_{F<\kappa}\int_F^\kappa f''(K)(K-F)dK \\ &= f(\kappa) + f'(\kappa)(F-\kappa) + \int_\kappa^\infty f''(K)(F-K)^+dK + \int_0^\kappa f''(K)(K-F)^+dK. \end{aligned} \quad (59)$$

Choosing $f(F) \equiv V(F,t)$ and setting $F = F_t$ and $\kappa = F_{t-}$ implies the result:

$$V(F_t,t) - V(F_{t-},t) - \frac{\partial V}{\partial F}(F_{t-},t)(F_t - F_{t-}) = \int_{F_{t-}}^\infty \frac{\partial^2 V}{\partial F^2}(K,t)(F-K)^+dK + \int_0^{F_{t-}} \frac{\partial^2 V}{\partial F^2}(K,t)(K-F)^+dK.$$

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