A METHODOLOGY FOR ASSESSING MODEL RISK AND ITS APPLICATION TO THE IMPLIED VOLATILITY FUNCTION MODEL

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Abstract

We propose a methodology for assessing model risk and apply it to the implied volatility function (IVF) model. This is a popular model among traders for valuing exotic options. Our research is different from other tests of the IVF model in that we reflect the traders' practice of recalibrating the model daily, or even more frequently, to the market prices of vanilla options. We find little evidence of model risk when the IVF model is used to price and hedge compound options. However, there is significant model risk when it is used to price and hedge some barrier options.

I. Introduction

In the 1980s and 1990s we have seen a remarkable growth in both the volume and the variety of the contracts that are traded in the over-the-counter derivatives market. Banks and other financial institutions rely on mathematical models for pricing and marking to market these contracts. As a result they have become increasingly exposed to what is termed "model risk". This is the risk arising from the use of an inadequate model. Some large financial institutions currently set aside reserves for model risk. This means that they defer recognition of part of their trading profits when the profits are considered to be dependent on the pricing model used.

Bank regulators currently require banks to hold capital for market risk and credit risk. The Basel Committee on Banking Supervision has indicated that it intends to expand the scope of the current capital adequacy framework so that it more fully captures the risks to which a bank is exposed.[1] In particular, it plans to require capital for operational risk, one important component of which is model risk. As the Basel Committee moves towards implementing its proposed new framework, banks are likely to come under increasing pressure to identify and quantify model risk.

As pointed out by Green and Figlewski (1999), an inadequate model may lead to a number of problems for a financial institution. It may cause contracts to be sold for too low a price or purchased for too high a price. It may also lead to a bad hedging strategy being used or cause market risk and credit risk measures to be severely in error.

Derman (1996) lists a number of reasons why a model may be inadequate. It may be badly specified; it may be incorrectly implemented; the model parameters may not be estimated correctly; and so on. In this paper we focus on model risk arising from the specification of the model.

The specification of the model is of most importance in the pricing and hedging of

[1] See Basel Committee for Banking Supervision (1999).

non-standard instruments such as exotic options and structured swaps. It is usually not a significant issue in the pricing and hedging of "plain vanilla" instruments such as European and American options on stock indices and currencies, and U.S. dollar interest rate caps and floors. A great deal of information on the way these instruments are priced by the market at any given time is available from brokers and other sources. As a result, the pricing and hedging of a new or an existing plain vanilla instrument is largely model independent.

We illustrate this point by considering European and American options on stock indices and currencies. Both are usually priced by traders using versions of the Black–Scholes model. This model implies that the implied volatility of an option is independent of strike price. However, it is now well established that implied volatilities of stock index and currency options vary systematically with strike price. Authors such as Rubinstein (1994) and Jackwerth and Rubinstein (1996) show that the implied volatilities of stock and stock index options exhibit a pronounced "skew". For options with a particular maturity, the implied volatility decreases as the strike price increases. For currency options, this skew becomes a "smile". For a given maturity, the implied volatility of an option on a foreign currency is a U-shaped function of the strike price. The implied volatility is lowest for an option that is at or close to the money. It becomes progressively higher as the option moves either in or out of the money.

Why do traders continue to use the Black–Scholes model and its extensions to price and hedge options when the model's assumptions about the underlying asset price are clearly wrong? The answer is that traders do not use the model to relate an option price to the price of the underlying asset. They use the model to ensure that one option is priced correctly relative to other options whose prices they can observe in the market. The model is a convenient interpolation tool. A typical approach when Black–Scholes is used is as follows. Data is collected each day from brokers and other sources on the Black–Scholes implied volatilities of a range of actively traded plain vanilla options on the underlying asset. Interpolation techniques are then used on the data to determine a complete function, known as the volatility surface, relating the Black–Scholes implied volatility for a plain vanilla option to its strike price and time to maturity. This volatility surface enables the price of any plain vanilla option to be calculated. Traders also use the model in a non-standard way when hedging. They hedge against possible changes in the volatility surface as well as against changes in the underlying asset price.

The key point here is that the pricing and hedging of plain vanilla options are not heavily dependent on the model used in markets where the options trade actively.[2] If a trader switched from a Black–Scholes model to another model (say, the constant elasticity of variance model) and used it in an analogous way to the Black–Scholes model, prices would change very little and hedging would be similar. The same is true for most other models that are used by traders in liquid markets.

One popular approach for testing a model is to compare prices produced by the model with market prices. Unfortunately, it is difficult to use this approach for assessing model risk. As we have just argued, if market prices are readily available for a particular instrument there is very little model risk in the way the instrument is priced and hedged. Model risk is primarily associated with non-standard instruments, and these are exactly the instruments for which we do not have market prices. Another approach for testing a model is to test directly the stochastic processes assumed by the model for market variables. This sometimes involves estimating the parameters of the stochastic process from sample data and then testing the process out of sample. Unfortunately, this approach is also of limited usefulness in the assessment of model risk. This is because in practice traders do not keep the parameters of the models they use constant.

The goal of the approaches just mentioned is to produce stationary models that fit the

[2] An exception would be a deep-out-of-the-money option where the strike price is outside the range strike prices normally traded.

market data as well as possible and are evolutionary realistic. In assessing model risk our goal is somewhat different. We are interested in estimating potential errors in a particular model as it is used in the trading room. Traders calibrate their models daily, or even more frequently, to market data. Any test that assumes the model is calibrated once to data and then used unchanged thereafter is liable to estimate model risk incorrectly.

Green and Figlewski (1999) provide one of the first tests of model risk. They recognize the important of reflecting the market's practice of recalibrating models daily. They consider the performance of traders who write options, pricing them with the Black–Scholes model. The volatility parameter used by a trader is assumed to be based on historical data and updated daily. Their simulated traders are behaving like real traders and not using the "pure" constant volatility version of the Black–Scholes model.

In practice traders like to calibrate a model to market prices rather than historical data, if at all possible. Our methodology for assessing model risk is different from Green and Figlewski and is designed to reflect this. We assume that prices in the market are governed by a plausible multi-factor no-arbitrage model that is quite different from, and more complex than, the model used by traders. We determine the parameters of the model by fitting it to representative market data. We compare the pricing and hedging performance of the traders' model with the pricing and hedging performance of the assumed true model. The market prices to which traders calibrate their model on an ongoing basis are generated by the assumed true model. If the pricing and hedging performance of the trader's model is good when compared to the assumed true model, we can be optimistic (but not certain) that there is little model risk. If the pricing and hedging performance is poor, there is evidence of model risk and grounds for setting aside reserves or using a different model.

We illustrate our approach by using it to assess model risk in the implied volatility function model, which is popular among traders for valuing exotic options. Section II describes the model and the way it is used by traders. Section III explains potential errors in the model. Section IV describes our pricing tests for compound options and barrier options. Section V describes our hedging tests. Conclusions are in Section VI.

II. The Implied Volatility Function Model

Many models have been developed to price exotic options using the Black–Scholes assumptions. Examples are Geske's (1979) model for pricing compound options, Merton's (1973) model for pricing barrier options, and Goldman et al's (1979) model for lookback options. If the implied volatilities of plain vanilla options were independent of strike price so that the Black–Scholes assumptions applied, it would be easy to use these models. Unfortunately, as pointed out in the previous section, the implied volatilities of equity and currency options are systematically dependent on strike price. It is difficult to use models based on the Black–Scholes assumptions for exotic options because there is no easy way of determining the appropriate volatility from the volatility surface for plain vanilla options.

Many more elaborate models than those based on the Black–Scholes assumptions have been developed. For example, Merton (1976) and Bates (1996) have proposed jump-diffusion models. Heston (1993), Hull and White (1987, 1988), and Stein and Stein (1995) have proposed models where volatility follows a stochastic process. These models are potentially useful for pricing exotic options because, when parameters are chosen appropriately, the Black–Scholes implied volatilities calculated from the models have a similar pattern to those observed in the market. However, the models are not widely used by traders. Most traders like to use a model for pricing exotic options that exactly matches the volatility surface calculated from plain vanilla options. Research by Rubinstein (1994), Derman and Kani (1994), Dupire (1994), and Andersen and Brotherton–Ratcliffe (1998) shows how a one-factor model with this property can be constructed by making volatility a function of the asset price and time. We will refer to this model as the implied volatility function (IVF) model.

The risk-neutral process followed by the asset price, S, in the IVF model is

$$\frac{dS}{S} = [r(t) - q(t)]dt + \sigma(S, t)dz \tag{1}$$

In this equation, the volatility of the asset price, $\sigma(S, t)$, is assumed to be a deterministic function of S and t, $r(t)$ is the risk-free rate, and $q(t)$ is the dividend yield. (When a currency is modeled it is treated as an asset providing a dividend yield equal to the foreign risk-free rate.) Derivatives dependent on the asset price satisfy the differential equation

$$\frac{\partial f}{\partial t} + [r(t) - q(t)]\frac{\partial f}{\partial S} + \frac{1}{2}\sigma(S,t)^2 S^2 \frac{\partial^2 f}{\partial S^2} = r(t)f \tag{2}$$

The variables $r(t)$ and $q(t)$ in equation (2) are set equal to the instantaneous forward risk-free rate and instantaneous forward dividend yield for a maturity t respectively. As shown by Dupire (1994) and Andersen and Brotherton–Ratcliffe (1998), the volatility function, $\sigma(S,t)$ can be determined analytically from the prices of European options with different strike prices and times to maturity. If we write $c_{\text{mkt}}(K,T)$ as the market price of a call option with strike K and maturity T, then:

$$\sigma(K,T)^2 = 2\frac{\partial c_{\text{mkt}}/\partial T + q(T)c_{\text{mkt}} + [r(T) - q(T)]K\partial c_{\text{mkt}}/\partial K}{K^2\partial^2 c_{\text{mkt}}/\partial K^2} \tag{3}$$

In Section I we outlined the way the Black–Scholes model is used in practice. The IVF model is used by traders in a similar way to Black–Scholes. It is used to price an exotic option consistently with plain vanilla options at one particular point in time. Each day, or even more frequently, the model's parameters are changed so that the model matches the current volatility surface and current information on the term structure and dividend yield. We refer to models that are used in this way as "Continual Recalibration" (CR) models. CR models can be contrasted with Single Calibration (SC) models which are calibrated once to either historical data or market prices and then used to relate an option price to the underlying stochastic variables through time. One of the features of derivatives markets appears to be that most models are developed by researchers as SC models and then used by traders as CR models.

There are important differences between the way CR and SC models are used for hedging. In an SC model, the user hedges against movements in all the variables that are assumed to be stochastic. In a CR model, the user typically hedges against movements in all the market prices that are used for calibration as well as against movements in the variables that are assumed to be stochastic. For example, when traders use either the Black–Scholes model or the IVF model, they hedge against movements in the volatility surface as well as against movements in the underlying asset price.

Dumas, Fleming and Whaley (1997) provide the most complete tests of the IVF model to date. They show that there are significant errors when the IVF model is fitted to the market at a particular time and then used to price options one week later. They also find that the difference between the observed and predicted option prices is larger for complex parameterizations of the volatility functions than for a constant volatility specification. Recently Rosenberg (2000) has proposed a model where the at-the-money implied volatility follows a process dependent on asset returns and other volatilities are a function of the at-the-money volatility. He carries out a similar test to Dumas, Fleming, and Whaley and shows that his model performs well when compared to particular cases of the IVF model.

These tests are interesting in that they show that the IVF model does not perform well as an SC model. Our research is different in that it is designed to assess the performance of the IVF model as it is used by traders; that is, as a CR model. We examine both the pricing and hedging performance of the model.

The CR-IVF model is quite different from the SC-IVF model. Producing a price from the CR-IVF model for an exotic option at a particular point in time involves two steps:

Step 1: Recalibrate the model so that $r(t)$, $q(t)$, and $\sigma(S, t)$ are consistent with current data on the volatility surface, interest rates, and dividends yields.

Step 2: Use equations (2) and (3) to value the exotic option

If we assume for simplicity that interest rates and dividends are deterministic, the CR-IVF model has the form

$$P = g(S, t, c_1, c_2, \ldots, c_n, \theta_1, \theta_2, \ldots, \theta_m) \tag{4}$$

where P is the price of an exotic option and g is a function. The variables c_i $(1 \leq i \leq n)$

are the prices of the European options used to define the volatility surface when the model is being calibrated. The variables θ_j ($1 \leq j \leq m$) are the constant parameters defining the option.

The CR-IVF model has two desirable properties. First, it satisfies the boundary conditions for P. Second, it prices all European options (and all linear combinations of European options) consistently with the market at all times. The SC-IVF model is used as a tool for creating a multifactor model with these properties. It is interesting to note that the SC-IVF model is a no-arbitrage model. However, the CR-IVF model, and most other CR models used by traders (including CR-Black-Scholes), are not arbitrage-free. Nevertheless, because they are used so widely in practice, it is important to examine their model risk.

III. Potential Errors in the CR-IVF Model

The CR-IVF model is designed so that it always values all European options correctly. This means that the risk-neutral probability distribution of the asset price at a future time is always correct. This in turn means that the CR-IVF model always correctly prices a derivative when its payoff is contingent on the asset price at only one time.

However, there is no guarantee that the CR-IVF model prices other derivatives correctly. Consider, for example, a compound option where the holder has the right at time T_1 to pay a prespecified amount of money, K_1, to enter into a European option with strike price, K_2, maturing at time T_2. The decision to exercise at time T_1 depends on the asset price at time T_1 and, possibly, other state variables. The payoff at time T_2 depends on the asset price at time T_2. The value of the option therefore depends on the joint probability distribution of the asset price at times T_1 and T_2. Because the joint probability distribution of two variables is not uniquely determined from their marginal distributions, we cannot be certain that prices given by the CR-IVF model are correct.

To express this more formally, define $\phi_n[t_1, t_2, \ldots, t_n]$ as the joint probability distribution of the asset price at times t_1, t_2, …, t_n. The CR-IVF model is designed so that $\phi_1(t_1)$ is correct for all t_1, but this does not ensure that $\phi_n[t_1, t_2, \ldots, t_n]$ is correct for $n > 1$. This point is also made by Britten–Jones and Neuberger (2000) who produce some interesting results characterizing the set of all continuous price processes that are consistent with a given set of option prices.

In the case of the compound option just considered $\phi_1(T_1)$ and $\phi_1(T_2)$ are correct, but this does not unambiguously determine $\phi_2(T_1, T_2)$. As a result there is potential for model risk. In the case of other derivatives, the dependence of the joint probability distribution of the underlying asset price at different times is quite complex. Consider a barrier option maturing at time T_N where the asset price is observed at times T_1, T_2, …, T_N for the purposes of determining whether the barrier has been hit. The price of the option depends on $\phi_N[T_1, T_2, \ldots, T_N]$. The CR-IVF model is designed so that $\phi_1(T_i)$

is correct for $1 \leq i \leq N$, but this does not unambiguously determine $\phi_N[T_1, T_2, \ldots, T_N]$. Again there is potential for model risk.

IV. Pricing Tests

As explained earlier our test of model risk involves assuming that market prices are determined by a no-arbitrage model that is more complex than, and quite different from, the model used by traders. To test the CR-IVF we assume the following two-factor stochastic volatility model:

$$\frac{dS}{S} = (r - q)dt + v dz_S \tag{5}$$

$$dv = \kappa(\theta - v)dt + \xi dz_v \tag{6}$$

In these equations z_S and z_v are Wiener processes with an instantaneous correlation ρ. The parameters κ, θ, and ξ are the mean-reversion rate, long-run average volatility, and standard deviation of the volatility, respectively, and are assumed to be constants. We also assume the spot rate, r, and the yield on the asset, q, are constants. The parameters of the model are chosen to minimize the root mean square error in matching the observed market prices for European options. The model is similar to the one proposed by Heston (1993) and has similar analytic properties.[3]

A valuation formula for the European call option price, $c_{\text{sv}}(S, v, t; K, T)$, in the model can be computed through the inversion of characteristic functions of random variables. It takes the form:

$$c_{\text{sv}}(S, v, t; K, T) = e^{-q(T-t)} S(t) F_1 - e^{-r(T-t)} K F_2 \tag{7}$$

[3] If $V = v^2$ is the variance rate, the model we assume implies that

$$dV = (\xi + 2\kappa\theta\sqrt{V} - 2\kappa V)\, dt + 2\xi\sqrt{V}\, dz_v$$

Heston's model is of the form:

$$dV = (\alpha - \beta V)\, dt + \gamma\sqrt{V}\, dz_v$$

More details on the model's analytic properties are available from the authors on request.

where F_1 and F_2 are integrals that can be evaluated efficiently using numerical procedures such as quadrature. More details on the model can be found in Schöbel and Zhu (1998).

We consider two sets of parameters for the stochastic volatility model. The first set is chosen by using a least squares procedure to provide as close a fit as possible to the volatility surface for the S&P 500 reported in Andersen and Brotherton–Ratcliffe (1998). It is

$$r = 5.9\%,\ q = 1.4\%,\ v(0) = 0.25,\ \kappa = 0.16,\ \theta = 0.3,\ \xi = 0.09, \text{ and } \rho = -0.79.$$

The second set is similarly chosen to find as close a fit as possible to the volatility surface for the U.S. dollar-Swiss franc exchange rate provided to us by a large U.S. investment bank. It is

$$r = 5.9\%,\ q = 3.5\%,\ v(0) = 0.13,\ \kappa = 0.11,\ \theta = 0.1,\ \xi = 0.04, \text{ and } \rho = 0.16.$$

The volatility surfaces given by the two parameter sets are shown in Tables 1 and 2. The volatility surfaces have the shapes mentioned earlier. The equity index volatility surface exhibits a volatility skew. The foreign currency volatility surface exhibits a volatility smile.

Our pricing tests of the CR-IVF model consist of the following steps:

Step 1: Price an exotic option using the best-fit stochastic volatility model. We denote this price by f_{true}.

Step 2: Fit the IVF model to the market prices of European call options that are given by the best-fit stochastic volatility model.

Step 3: Use the IVF model to price the same exotic option. We denote this price by f_{ivf}.

Step 4: Use the Black–Scholes assumptions to price the exotic option. We denote this price by f_{bs}.

Step 5: Compare f_{true}, f_{ivf}, and f_{bs}.

We calculate the market prices of European call options, c_{mkt}, using equation (7). We fit the IVF model to these prices by calculating $\partial c_{\text{mkt}}/\partial t$, $\partial c_{\text{mkt}}/\partial K$, and $\partial^2 c_{\text{mkt}}/\partial K^2$ from equation (7) and then using equation (3).

We consider two types of exotic options: a call-on-call compound option and a knock-out barrier option. We use Monte Carlo simulation with 300 time steps and 100,000 trials to estimate the prices of these options for the stochastic volatility model.[4] For this purpose, equations (5) and (6) are discretized to

$$\ln \frac{S_{i+1}}{S_i} = \left(r - q - \frac{v_i^2}{2}\right)\Delta t + v_i \epsilon_1 \sqrt{\Delta t} \tag{8}$$

$$v_{i+1} - v_i = \kappa(\theta - v_i)\Delta t + \xi \epsilon_2 \sqrt{\Delta t} \tag{9}$$

where Δt is the length of the Monte Carlo simulation time step, S_i and v_i are the asset price and its volatility at time $i\Delta t$, and ϵ_1 and ϵ_2 are random samples from two unit normal distributions with correlation, ρ.

We estimate the prices given by the IVF model from equation (2) using the implicit Crank-Nicholson finite difference method described in Andersen and Brotherton–Ratcliffe (1998). This involves constructing a 120 $\times$ 70 rectangular grid of points in (x, t)-space, where $x = \ln S$. The grid extends from time zero to the maturity of the exotic option, T_{mat}. Define x_{min} and x_{max} as the lowest and highest x-values considered on the grid. (We explain how these are determined later.) Boundary conditions determine the values of the exotic option on the $x = x_{\text{max}}$, $x = x_{\text{min}}$ and $t = T_{\text{mat}}$ edges of the grid. The differential equation (2) enables relationships to be established between the values of the exotic option at the nodes at the ith time point and its values at the nodes at the $(i+1)$th time point. These relationships are used in conjunction with boundary conditions to determine the value of the exotic option at all interior nodes of the grid and its value at the nodes at time zero.

A. Compound Options

Our first test of the IVF model uses a call-on-call compound option. This is an option where the holder has the right at time T_1 to pay K_1 and obtain a European call option

[4] To reduce the variance of the estimates, we use the antithetic variable technique described in Boyle (1977).

to purchase the asset for a price K_2 at time T_2 ($T_2 > T_1$). When using Monte Carlo simulation to calculate f_{true}, each trial involves using equations (8) and (9) to calculate the asset price and its volatility at time T_1. It is not necessary to simulate beyond time T_1 because the value of a European call option with strike price K_2 and maturity T_2 can be calculated at time T_1 using equation (7). Define $S_{1,j}$ and $v_{1,j}$ as the asset price and volatility at time T_1 on the jth trial, and $w_{1,j}$ as the value at time T_1 of a call option with strike price K_2 maturing at T_2 for the jth trial. It follows that

$$w_{1,j} = c_{\text{sv}}(S_{1,j}, v_{1,j}, T_1, K_2, T_2)$$

The estimate of the true value of the option given by Monte Carlo simulation is:

$$f_{\text{true}} = \frac{e^{-rT_1}}{N} \sum_{j=1}^{N} \max(w_{1,j} - K_1,\, 0)$$

We calculate the IVF price for the compound option by building the finite difference grid out to time T_2. Between times T_1 and T_2, we use the grid to calculate the price, w, of a European call option with strike price K_2 maturing at time T_2. This enables the value of the compound option at the nodes at time T_1 to be calculated as $\max(w - K_1, 0)$. We then use the part of the grid between time zero and time T_1 to calculate the value of the compound option at time zero. We set $x_{\min} = \ln S_{\min}$ and $x_{\max} = \ln S_{\max}$ where $S_{\min}$ and $S_{\max}$ are very high and very low asset prices, respectively. The boundary conditions we use are:

$$w = \max(e^x - K_2, 0) \text{ when } t = T_2$$

$$w = 0 \text{ when } x = x_{\min} \text{ and } T_1 \le t \le T_2$$

$$w = e^x - K_2 e^{-r(T_2 - t)} \text{ when } x = x_{\max} \text{ and } T_1 \le t \le T_2$$

$$f_{\text{ivf}} = 0 \text{ when } x = x_{\min} \text{ and } 0 \le t \le T_1$$

$$f_{\text{ivf}} = e^x - K_2 e^{-r(T_2 - t)} - K_1 e^{-r(T_1 - t)} \text{ when } x = x_{\min} \text{ and } 0 \le t \le T_1$$

The value of a compound option using the Black–Scholes assumptions was first produced by Geske (1979). Geske shows that at time zero:

$$f_{\text{bs}} = S(0) e^{-qT_2} M(a_1, b_1; \sqrt{T_1/T_2}) - K_2 e^{-rT_2} M(a_2, b_2; \sqrt{T_1/T_2}) - e^{-rT_1} K_1 N(a_2)$$

where

$$a_1 = \frac{\ln[S(0)/S^*] + (r - q + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \qquad a_2 = a_1 - \sigma\sqrt{T_1}$$

$$b_1 = \frac{\ln[S(0)/K_2] + (r - q + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, \qquad b_2 = b_1 - \sigma\sqrt{T_2}$$

and $M(a, b; \rho)$, is the cumulative probability in a standardized bivariate normal distribution that the first variable is less than a and the second variable is less than b when the coefficient of correlation between the variables is ρ. The variable S^* is the asset price at time T_1 for which the price at time T_1 of a European call option with strike price K_2 and maturity T_2 equals K_1. If the actual asset price is above S^* at time T_1, the first option will be exercised; if it is not above S^*, the compound option expires worthless. In computing f_{bs} we set σ equal to the implied volatility of a European option maturing at time T_2 with a strike price of K_2.

Table 3 shows f_{true} and the percentage errors when the option price is approximated by f_{ivf} and f_{bs} for the case where $T_1 = 1$, $T_2 = 2$, and K_2 equals the initial asset price. It considers a wide range of values of K_1. The table shows that the IVF model works very well. For compound options where the true price is greater than 1% of the initial asset price, the IVF price is within 2% of the true price. When very high strike prices are used with Parameter Set II this percentage error is higher, but this is because the true price of the compound option is very low. Measured as a percent of the initial asset price the absolute pricing error of the IVF model is never greater than 0.08%.

The Black–Scholes model, on the other hand, performs quite badly. For high values of the strike price, K_1, it significantly overprices the compound option in the case of the stock index data and significantly underprices it in the case of the foreign currency data. The reason is that, when K_1 is high, the first call option is exercised only when the asset price is very high at time T_1. Consider first the stock index data. As shown in Table 1, the implied volatility is a declining function of the strike price. (This is the volatility skew phenomenon for a stock index described in Section I). As a result the probability distribution of the asset price at time T_1 has a more heavy left tail and a less heavy right tail than a lognormal distribution when the latter is calculated using the at-the-money volatility, and very high asset prices are much less likely than they are under the Black–Scholes model. This means that the first option is much more likely to be exercised at time T_1 in the Black–Scholes world than in the assumed true world. Consider next the foreign currency data. As shown in Table 2, the implied volatility is a U-shaped function. (This is the volatility smile phenomenon for a currency described in Section I.) The results in the probability distribution of the asset price having heavier left and right tails than a lognormal distribution when the latter is calculated using the at-the-money volatility, and very high asset prices are much more likely than they are under the Black–Scholes model. This means that the first option is much less likely to be exercised in the Black–Scholes world than in the assumed true world.

Traders sometimes try to make the Black–Scholes model work for compound options by adjusting the volatility. Sometimes they use two different volatilities, one for the period between time zero and time T_1 and the other for the period between time T_1 and time T_2. There is of course some volatility (or pair of volatilities) that will give the correct price for any given compound option. But the price of a compound option given by the Black-Scholes model is highly sensitive to the volatility and any procedure that involves estimating the "correct" volatility is dangerous and liable to give rise to significant errors.

Based on the tests reported here and other similar tests we have carried out, the IVF model is a big improvement over the Black–Scholes model when compound options are priced. There is very little evidence of model risk. This is encouraging, but of course it provides no guarantee that the model is also a proxy for all more complicated multifactor models.

B. Barrier Options

The second exotic option we consider is a knock-out barrier call option. This is a

European call option with strike price K and maturity T that ceases to exist if the asset price reaches a barrier level, H. When the barrier is greater than the initial asset price, the option is referred to as an up-and-out call; when the barrier is less than the initial asset price, it is referred to as a down-and-out call.

When using Monte Carlo simulation to calculate f_{true}, each trial involved using equations (8) and (9) simulate a path for the asset price between time zero and time T. For an up-and-out (down-and-out) option, if for some i, the asset price is above (below) H at time $i\Delta t$ on the jth trial the payoff from the barrier option is set equal to zero on that trial. Otherwise the payoff from the barrier option is $\max[S(T) - K, 0]$ at time T. The estimate of f_{true} is the arithmetic mean of the payoffs on all trials discounted from time T to time zero at rate r.[5]

We calculate the IVF price for the barrier option by building the finite difference grid out to time T. In the case of an up-and-out option, we set $x_{\max} = \ln(H)$ and $x_{\min} = \ln S_{\min}$ where $S_{\min}$ is a very low asset price; in the case of a down-and-out option, we set $x_{\min} = \ln(H)$ and $x_{\max} = \ln S_{\max}$ where $S_{\max}$ is a very high asset price. For an up-and-out call option, the boundary conditions are:

$$f_{\text{ivf}} = \max(e^x - K_2, 0) \text{ when } t = T$$

$$f_{\text{ivf}} = 0 \text{ when } x \geq \ln(H) \text{ and } 0 \leq t \leq T$$

$$f_{\text{ivf}} = 0 \text{ when } x = x_{\min} \text{ and } 0 \leq t \leq T$$

For a down-and-out call, the boundary conditions are similar except that

$$f_{\text{ivf}} = e^x - K_2 e^{-r(T-t)} \text{ when } x = x_{\max}$$

The value of knock-out options using the Black–Scholes assumptions was first produced by Merton (1973). He showed that at time zero, the price of a down-and-out call

[5] To improve computational efficiency we applied the correction for discrete observations discussed in Broadie, Glasserman, and Kou (1997).

option is

$$f_{\rm bs} = S(0)N(d_1)e^{-qT} - KN(d_2)e^{-rT} - S(0)e^{-qT}[H/S(0)]^{2\lambda}N(y)$$

$$+Ke^{-rT}[H/S(0)]^{2\lambda-2}N(y-\sigma\sqrt{T})$$

and that the price of an up-and-out call is

$$f_{\rm bs} = S(0)e^{-qT}[N(d_1) - N(x_1)] - Ke^{-rT}[N(d_2) - N(x_1 - \sigma\sqrt{T})]$$

$$+S(0)e^{-qT}[H/S(0)]^{2\lambda}[N(-y) - N(-y_1)]$$

$$-Ke^{-rT}[H/S(0)]^{2\lambda-2}[N(-y+\sigma\sqrt{T}) - N(-y_1+\sigma\sqrt{T})]$$

where

$$\begin{aligned}
\lambda &= \frac{r-q+\sigma^2/2}{\sigma^2} \\
y &= \frac{\ln\{H^2/[S(0)K]\}}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\
x_1 &= \frac{\ln[S(0)/H]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\
x_2 &= \frac{\ln[H/S(0)]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\
d_1 &= \frac{\ln[S(0)/K] + (r-q+\sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}$$

and N is the cumulative normal distribution function. In computing $f_{\rm bs}$ we set σ equal to the implied volatility of a regular European call option with strike price K maturing at time T.

Tables 4 and 5 show $f_{\rm true}$ and the percentage errors when the option is approximated by $f_{\rm ivf}$ and $f_{\rm bs}$ for the cases where K are 90% and 100% of the initial asset price and the time to maturity is two years. We consider a wide range of values for the barrier H. (When $H > 100$ the option is an up-and-out call; when $H < 100$ it is a down-and-out call.) A comparison of Table 3 with Tables 4 and 5 shows that the IVF model does not perform as well for barrier options as it does for compound options. For some values of the barrier, errors are high in both absolute terms and percentage terms in both the stock index and foreign currency cases

To illustrate the reasons for the results, consider the equity index case when the strike price is 100 and the barrier is 130. The Black-Scholes price is a significantly convex function of volatility. As shown by Hull and White (1987), when there is zero correlation between the asset price and volatility, the stochastic volatility price of a barrier option is its Black–Scholes price integrated over the probability distribution of the average variance rate during the life of the option. The convexity therefore leads to the value of the option increasing as the volatility of the volatility increases in the zero correlation case. For the equity index, the correlation between stock price and volatility is negative rather than zero. This increases the value of the option still further because high stock prices tend to be associated with low volatilities making it less likely that the barrier will be hit.

These arguments explain why the Black–Scholes price of the option we are considering is 38.61% less than the stochastic volatility price. Consider next the IVF price. The IVF model does incorporate a negative correlation between stock price and volatility. However, it is a one-factor model and does not lead to as wide a range of high and low volatility outcomes as the stochastic volatility model. As a result it does not fully reflect the impact of the convexity in Figure 1 and gives a lower price than the stochastic volatility model.

Other results in Tables 5 and 6 can be explained using similar arguments. We conclude from these and other similar tests that there is significant model risk when the IVF model is used to price barrier options for some sets of parameter values. This is an important result. Barrier option are the probably more actively traded than any other type of exotic option.

V. Hedging Using the CR-IVF Model

As already mentioned, when traders use the CR-IVF model, they hedge against changes in the volatility surface as well as against changes in the asset price. The model has the form shown in equation (4). Traders calculate $\partial P/\partial c_i$ for $1 \leq i \leq n$ (or equivalent partial derivatives involving attributes of the volatility surface) as well as $\partial P/\partial S$ and attempt to combine their positions in exotic options with positions in the underlying asset and positions in European options to create a portfolio that is riskless when it is valued using the CR-IVF model.[6] They means that they create a portfolio whose value, Π, (as measured by the CR-IVF model) satisfies

$$\frac{\partial \Pi}{\partial c_i} = 0$$

for $(1 \leq i \leq n)$ and

$$\frac{\partial \Pi}{\partial S} = 0$$

Our test of the hedging effectiveness of the CR-IVF model is analogous to our test of its pricing effectiveness. We assume that the fitted two-factor stochastic volatility model in equations (5) and (6) gives the true price of an exotic option. This model has two underlying stochastic variables: S, the asset price and v, the volatility. We test whether that the CR-IVF model gives reasonable estimates of the sensitivities of the exotic option price changes to S and v. We calculate numerically the partial derivative of exotic option prices with respect to each of S and v for both the stochastic volatility model and the CR-IVF model. The partial derivative with respect to S is the delta, Δ, and the partial derivative with respect to v is the vega, $\mathcal{V}$, of the exotic option.

In order to compute Δ, we increase the spot price by 1%, and compute the price

[6] Given the nature of the CR-IVF model the partial derivatives of P with respect to S and the c_i cannot be calculated analytically. They must be calculated by perturbing the stock price and each of the n option prices in turn, recalibrating the model, and observing the effect on the exotic option price, P.

changes from both the true model and the CR-IVF model.[7] To compute vega we increase the initial instantaneous volatility by 1% and compute price changes for both the true model and the CR-IVF model. Note that the calculation of delta for the CR-IVF model involves calibrating the model twice to the option prices given by the true model, once before and once after S has been perturbed. The same is true of vega.

Our hedging results for the compound options considered in Table 3 are shown in Table 6. The table shows that using the CR-IVF model for hedging gives good results in a world described by the two-factor stochastic volatility model. Our hedging results for the barrier option considered in Table 4 are shown in Table 7. The percentage errors in the deltas calculated by the CR-IVF model are similar to the percentage pricing errors. The vegas calculated for the CR-IVF model are quite often markedly different from the vegas calculated using the stochastic volatility model. This is indication that volatility hedges created using the CR-IVF model for barrier options may not be effective. The tests for the barrier options in Table 5 and others we considered produced similar results.

[7] We also computed delta for the SC-IVF model. These are not reported, but are slightly worse than the deltas from the CR-IVF model.

VI. Summary

It is important that tests of model risk reflect the way models are actually used by traders. Researchers and traders often use models differently. A model when it is developed by researchers is usually a SC (single calibration) model. When the same model is used by researchers it is a CR (continual recalibration) model. The researcher-developed SC models are usually arbitrage-free. The corresponding CR models used by traders are not in general arbitrage free. This paper is one of the first attempts to assess the performance of a CR model.

The CR-IVF model has the attractive feature that it always matches the prices of European options. This means that the unconditional probability distribution of the underlying asset price at all future times is always correct. An exotic option, whose payoff is contingent on the asset price at just one time is always correctly priced by the CR-IVF model. Unfortunately, many exotic options depend on the joint probability distribution of the asset price at two or more times. There is no guarantee that the CR-IVF model will provide good pricing and hedging for these instruments.

In this paper we examine the model risk in the CR-IVF model by fitting a stochastic volatility model to market data and then comparing the prices of compound options and barrier options with those given by the IVF model. We find that the CR-IVF model gives reasonably good results for compound options. The results for barrier options are much less satisfactory. The CR-IVF model does not recover enough aspects of the dynamic features of the asset price process to give reasonably accurate prices for some combinations of the strike price and barrier level. The hedge parameters produced by the model also sometimes have large errors.

There are a number of possible extensions of the research in this paper. The performance of the IVF model could be tested using other “true” models. Also the same methodology could be used for other models used by traders. For example it could be used to test model risk in one-factor CR-term structure models.

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TABLE 1
Volatility Surface for Stock Index Options. K=strike price as a percent of the initial asset price; T=time to maturity.

	K											
T	70.00	80.00	85.00	90.00	95.00	100.00	105.00	110.00	115.00	120.00	130.00	140.00
0.175	29.67	28.06	27.31	26.58	25.87	25.16	24.47	23.79	23.13	22.48	21.29	21.35
0.425	29.71	28.20	27.47	26.76	26.07	25.39	24.71	24.05	23.40	22.77	21.56	20.49
0.695	29.82	28.34	27.64	26.95	26.27	25.61	24.95	24.31	23.68	23.06	21.88	20.81
0.940	29.91	28.47	27.78	27.11	26.45	25.80	25.17	24.54	23.93	23.33	22.17	21.11
1.000	29.94	28.50	27.82	27.15	26.49	25.85	25.22	24.60	23.99	23.39	22.24	21.18
1.500	30.11	28.74	28.08	27.45	26.82	26.21	25.61	25.02	24.45	23.88	22.78	21.76
2.000	30.27	28.95	28.32	27.71	27.12	26.54	25.97	25.41	24.86	24.32	23.28	22.30
3.000	30.53	29.31	28.73	28.18	27.63	27.10	26.58	26.08	25.58	25.09	24.15	23.26
4.000	30.74	29.61	29.08	28.56	28.06	27.57	27.10	26.63	26.18	25.74	24.89	24.07
5.000	30.91	29.86	29.36	28.88	28.42	27.97	27.53	27.11	26.69	26.29	25.51	24.76

The parameters for the stochastic volatility model used in generating this table are: $r = 5.9\%$, $q = 1.4\%$, $v_0 = 0.25$, $\kappa = 0.16$, $\theta = 0.3$, $\xi = 0.09$, and $\rho = -0.79$.

TABLE 2
Volatility Surface for Foreign Currency Options. K=strike price as a percent of the initial asset price; T=time to maturity.

	K											
T	70.00	80.00	85.00	90.00	95.00	100.00	105.00	110.00	115.00	120.00	130.00	140.00
0.175	23.03	14.56	12.85	12.73	12.75	12.83	12.96	13.13	13.32	13.56	16.16	21.10
0.425	14.97	12.90	12.77	12.72	12.74	12.81	12.93	13.09	13.27	13.46	13.88	14.50
0.695	13.44	12.89	12.77	12.71	12.72	12.79	12.90	13.06	13.23	13.42	13.81	14.20
0.940	13.32	12.89	12.76	12.70	12.71	12.77	12.88	13.02	13.19	13.37	13.76	14.14
1.000	13.31	12.89	12.76	12.70	12.70	12.76	12.87	13.02	13.18	13.36	13.74	14.12
1.500	13.30	12.89	12.76	12.69	12.68	12.73	12.82	12.96	13.11	13.29	13.65	14.02
2.000	13.30	12.89	12.76	12.69	12.67	12.70	12.78	12.90	13.05	13.21	13.56	13.92
3.000	13.30	12.91	12.77	12.68	12.65	12.66	12.72	12.81	12.94	13.08	13.40	13.73
4.000	13.31	12.92	12.78	12.69	12.64	12.63	12.67	12.74	12.84	12.97	13.25	13.56
5.000	13.31	12.93	12.80	12.70	12.64	12.62	12.63	12.69	12.77	12.87	13.13	13.41

The parameters for the stochastic volatility model used in generating this table are: $r = 5.9\%$, $q = 3.5\%$, $v_0 = 0.1285$, $\kappa = 0.1090$, $\theta = 0.10$, $\xi = 0.0376$, and $\rho = 0.1548$.

TABLE 3
Numerical Results for Compound Options

	Stock Index Data (Table 1)			Foreign Currency Data (Table 2)		
K_1	True Price	IVF % Error	BS % Error	True Price	IVF % Error	BS % Error
3.45	15.45	-0.05	0.43	6.24	-0.95	-0.72
5.90	13.60	-0.14	1.23	4.82	-1.64	-1.76
8.35	11.93	-0.25	2.47	3.70	-1.67	-3.13
10.80	10.42	-0.36	4.14	2.83	-1.73	-4.78
13.25	9.07	-0.45	6.28	2.14	-1.63	-6.77
15.70	7.85	-0.54	8.99	1.62	-1.42	-9.22
18.15	6.76	-0.53	12.31	1.21	-1.07	-12.13
20.60	5.79	-0.59	16.34	0.90	-0.77	-15.49
23.05	4.92	-0.47	21.14	0.67	-1.04	-19.49
25.50	4.16	-0.36	26.93	0.50	-0.80	-24.05
27.95	3.50	-0.26	33.77	0.37	0.81	-28.72
30.40	2.92	-0.14	41.89	0.27	1.47	-33.82
32.85	2.42	0.00	51.39	0.20	1.00	-39.00
35.30	1.99	0.25	62.58	0.15	2.74	-43.84
37.75	1.62	0.62	75.85	0.11	4.67	-49.53
40.20	1.32	1.06	91.48	0.08	5.13	-55.13
42.65	1.06	1.42	110.02	0.06	7.02	-59.65
45.10	0.84	1.68	131.75	0.04	7.14	-64.29

The table shows the true price as a percent of the initial asset price, and the percentage error when this is approximated using the IVF model and the Black–Scholes model. The maturity of the first option, T_1, is 1 year; the maturity of the second option, T_2, is 2 years; the second strike price, K_2 equals the initial asset price; the first strike price K_1 is shown in the table as a percentage of the initial asset price.

TABLE 4
Numerical Results for Knock-Out Barrier Options when the Strike Price is 90% of the Initial Asset Price.

	Stock Index Data (Table 1)			Foreign Currency Data (Table 2)		
H	True Price	IVF % Error	BS % Error	True Price	IVF % Error	BS % Error
50	24.00	0.09	0.18	14.94	0.07	0.02
60	23.77	-0.06	0.64	14.94	0.04	0.02
70	22.69	0.00	1.67	14.93	0.09	0.05
80	19.74	-0.42	0.75	14.72	0.33	0.37
90	12.97	-0.69	-3.44	12.08	-0.75	-0.57
92	10.95	-0.74	-4.67	10.58	-0.51	-0.09
94	8.72	-1.20	-6.54	8.67	-0.56	0.10
96	6.18	-1.84	-8.75	6.32	-1.32	-0.08
98	3.36	-4.08	-12.84	3.42	-1.44	0.50
104	0.03	-26.60	-17.49	0.26	-21.65	-26.61
106	0.06	-25.91	-20.02	0.52	-18.28	-22.92
108	0.11	-26.61	-23.54	0.87	-16.81	-20.16
110	0.18	-24.16	-24.84	1.32	-14.33	-17.43
120	1.01	-18.54	-31.40	4.42	-6.62	-4.37
130	2.90	-13.85	-36.74	7.74	-2.71	3.58
140	5.91	-10.39	-40.49	10.41	-1.37	6.40
150	9.66	-8.15	-42.35	12.19	-0.52	6.81

The table shows the true price as a percent of the initial asset price, and the percentage error when this is approximated using the IVF model and the Black–Scholes model. The barrier, H, is shown in the table as a percent of the initial asset price. The time to maturity is two years.

TABLE 5
Numerical Results for Knock-Out Barrier Options when the Strike Price Equals the Initial Asset Price.

	Stock Index Data (Table 1)			Foreign Currency Data (Table 2)		
H	True Price	IVF % Error	BS % Error	True Price	IVF % Error	BS % Error
50	18.40	0.07	0.13	8.93	0.01	0.02
60	18.29	-0.05	0.54	8.93	0.01	0.02
70	17.61	0.11	2.12	8.93	0.02	0.04
80	15.61	-0.25	2.79	8.86	0.39	0.42
90	10.55	-0.54	0.52	7.72	0.48	0.37
92	8.96	-0.59	-0.37	6.91	1.14	1.10
94	7.18	-0.97	-1.91	5.84	1.11	1.10
96	5.12	-1.64	-3.91	4.39	0.30	0.64
98	2.80	-3.90	-7.96	2.45	0.04	0.88
104	0.00	-42.41	-9.93	0.01	-46.41	-34.04
106	0.00	-36.50	-19.52	0.03	-33.51	-29.92
108	0.01	-40.69	-28.23	0.09	-30.36	-25.62
110	0.03	-32.00	-27.10	0.19	-24.08	-21.73
120	0.34	-24.21	-33.29	1.44	-10.56	-4.18
130	1.39	-18.08	-38.61	3.41	-4.36	6.64
140	3.40	-13.40	-42.20	5.27	-2.33	10.12
150	6.20	-10.18	-43.88	6.62	-0.90	10.38

The table shows the true price as a percent of the initial asset price, and the percentage error when this is approximated using the IVF model and the Black–Scholes model. The barrier, H, is shown in the table as a percent of the initial asset price. The time to maturity is two years.

TABLE 6
Price Sensitivities for Compound Options

	Stock Index Data (Table 1)				Foreign Currency Data (Table 2)			
K_1	Δ	Δ_{ivf}	$\mathcal{V}$	$\mathcal{V}_{\text{ivf}}$	Δ	Δ_{ivf}	$\mathcal{V}$	$\mathcal{V}_{\text{ivf}}$
3.45	0.691	0.699	0.398	0.398	0.541	0.556	0.436	0.440
5.90	0.667	0.676	0.404	0.407	0.473	0.496	0.421	0.435
8.35	0.637	0.641	0.409	0.411	0.402	0.392	0.394	0.393
10.80	0.603	0.606	0.412	0.414	0.335	0.332	0.362	0.362
13.25	0.566	0.568	0.412	0.410	0.274	0.272	0.324	0.325
15.70	0.528	0.533	0.408	0.411	0.219	0.218	0.283	0.284
18.15	0.489	0.498	0.402	0.405	0.174	0.170	0.244	0.244
20.60	0.449	0.446	0.392	0.389	0.136	0.131	0.207	0.207
23.05	0.410	0.405	0.380	0.377	0.106	0.106	0.172	0.173
25.50	0.371	0.367	0.364	0.361	0.081	0.088	0.142	0.145
27.95	0.333	0.330	0.346	0.343	0.062	0.065	0.116	0.119
30.40	0.296	0.296	0.325	0.328	0.048	0.048	0.094	0.096
32.85	0.261	0.261	0.303	0.306	0.036	0.036	0.076	0.077
35.30	0.228	0.227	0.279	0.281	0.027	0.029	0.060	0.062
37.75	0.197	0.199	0.255	0.256	0.020	0.022	0.047	0.050
40.20	0.170	0.175	0.231	0.231	0.015	0.016	0.037	0.039
42.65	0.144	0.140	0.206	0.209	0.011	0.012	0.029	0.031
45.10	0.121	0.123	0.182	0.185	0.008	0.009	0.023	0.024

Δ and Δ_{ivf} are the rate of change of the option price with the asset price for the true model and the CR-IVF model, respectively. $\mathcal{V}$ and $\mathcal{V}_{\text{ivf}}$ are the change in the option price (measured as a percent of the initial stock price) for a 1% change in the initial instantaneous volatility for the true model and the CR-IVF model, respectively. The maturity of the first option, T_1, is 1 year; the maturity of the second option, T_2, is 2 years; the second strike price, K_2 equals the initial asset price; the first strike price K_1 is shown in the table as a percentage of the initial asset price.

TABLE 7
Price sensitivities for Knock-Out Barrier Options when the Strike Price is 90% of the Initial Asset Price.

	Stock Index Data (Table 1)				Foreign Currency Data (Table 2)			
K_1	Δ	Δ_{ivf}	$\mathcal{V}$	$\mathcal{V}_{\text{ivf}}$	Δ	Δ_{ivf}	$\mathcal{V}$	$\mathcal{V}_{\text{ivf}}$
50	0.786	0.788	0.322	0.318	0.781	0.782	0.298	0.299
60	0.793	0.799	0.254	0.283	0.781	0.785	0.297	0.299
70	0.836	0.836	0.143	0.189	0.783	0.785	0.293	0.297
80	0.931	0.941	-0.034	0.035	0.807	0.807	0.244	0.217
90	1.186	1.182	-0.064	-0.107	1.077	1.079	-0.213	-0.132
92	1.269	1.260	-0.194	-0.112	1.185	1.198	-0.200	-0.188
94	1.326	1.356	-0.158	-0.110	1.327	1.320	-0.187	-0.212
96	1.457	1.429	-0.180	-0.093	1.460	1.464	-0.198	-0.191
98	1.442	1.530	-0.073	-0.058	1.657	1.634	-0.094	-0.120
104	-0.007	-0.007	-0.003	-0.001	-0.061	-0.052	-0.047	-0.034
106	-0.011	-0.008	-0.002	-0.003	-0.080	-0.071	-0.090	-0.070
108	-0.015	-0.011	-0.009	-0.006	-0.095	-0.081	-0.150	-0.116
110	-0.022	-0.017	-0.015	-0.010	-0.100	-0.096	-0.212	-0.174
120	-0.070	-0.055	-0.084	-0.063	-0.039	-0.045	-0.502	-0.486
130	-0.116	-0.116	-0.233	-0.195	0.141	0.118	-0.586	-0.614
140	-0.164	-0.162	-0.452	-0.403	0.333	0.318	-0.514	-0.534
150	-0.148	-0.148	-0.044	-0.039	0.488	0.487	-0.327	-0.362

Δ and Δ_{ivf} are the rate of change of the option price with the asset price for the true model and the CR-IVF model, respectively. $\mathcal{V}$ and $\mathcal{V}_{\text{ivf}}$ are the change in the option price (measured as a percent of the initial stock price) for a 1% change in the initial instantaneous volatility for the true model and the CR-IVF model, respectively. The barrier, H, is shown as a percent of the initial asset price. The time to maturity is two years.