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Conditional Value-at-Risk: Optimization Algorithms and Applications

By Stanislav Uryasev, Ph.D.

Value-at-Risk (VaR), a widely used performance measure, answers the question: what is the maximum loss with a specified confidence level? A description of various methodologies for the modeling of VaR can be seen, along with related resources, at the Web site http://www.gloriamundi.org. For the most part, approaches to calculating VaR rely on linear approximation of risks and assume the joint normal (or log-normal) distribution of the underlying market parameters. Probably the most influential contribution in this field has been RiskMetrics methodology [9]. Although VaR is a very popular measure of risk, it has undesirable properties [2] such as lack of sub-additivity, i.e., VaR of a portfolio with two instruments may be greater than the sum of individual VaRs of these two instruments. Also, VaR is difficult to optimize when calculated using scenarios. In this case, VaR is non-convex (see definition of convexity in [10]), non-smooth as a function of positions, and it has multiple local extrema.

An alternative measure of losses, with more attractive properties, is Conditional Value-at-Risk (CVaR), which is also called Mean Excess Loss, Mean Shortfall, or Tail VaR. CVaR is a more consistent measure of risk since it is sub-additive and convex [2]. Moreover, as it was shown recently [12], it can be optimized using linear programming (LP) and nonsmooth optimization algorithms, which allow handling portfolios with very large numbers of instruments and scenarios. Numerical experiments indicate that the minimization of CVaR also leads to near optimal solutions in VaR terms because CVaR is always greater than or equal to VaR. Moreover, when the return-loss distribution is normal, these two measures are equivalent [12], i.e., they provide the same optimal portfolio.

CVaR can be used in conjunction with VaR and is applicable to the estimation of risks with non-symmetric return-loss distributions. Although CVaR has not become a standard in the finance industry, it is likely to play a major role as it currently does in the insurance industry. Similar to the Markowitz [5] mean-variance approach, CVaR can be used in return-risk analyses. For instance, we can calculate a portfolio with a specified return and minimal CVaR. Alternatively, we can constrain CVaR and find a portfolio with maximal return, see [8]. Also, rather than constraining the variance, we can specify several CVaR constraints simultaneously with various confidence levels (thereby shaping the loss distribution), which provides a flexible and powerful risk management tool.

Several case studies showed that risk optimization with the CVaR performance function and constraints can be done for large portfolios and a large number of scenarios with relatively small computational resources. For instance, a problem with 1,000 instruments and 20,000 scenarios can be optimized on a 300 MHz PC in less than one minute using the CPLEX LP solver. A case study on the hedging of a portfolio of options using the CVaR minimization technique is included in [12]. This problem was first studied at Algorithmics, Inc. with the minimum expected regret approach [6]. Also, the CVaR minimization approach was applied to credit risk management of a portfolio of bonds [1]. This portfolio was put together by several banks to test various credit risk modeling techniques. Earlier, the minimum expected regret optimization technique was applied to the same portfolio at Algorithmics, Inc.[7]; we have used the same set of scenarios to test the minimum CVaR technique. A case study on optimization of a portfolio of stocks with CVaR constraints is included in [8]. The reader interested in other applications of optimization techniques in the finance area can find relevant papers in [14].

1. Approach

This section outlines the approach suggested in [12] for simultaneous minimization of CVaR and calculation of VaR. The next section discusses how to extend this idea to problems with CVaR constraints.

Let $f(\boldsymbol{x},\boldsymbol{y})$ be a loss function depending upon the decision vector $\boldsymbol{x}$ and a random vector $\boldsymbol{y}$. The decision vector $\boldsymbol{x}$ belongs to a feasible set of portfolios, $\boldsymbol{X}$, satisfying imposed requirements. For example, we may consider portfolios with non-negative positions (short positions are not allowed) and an expected return greater than 10%.

Example 1. Two Instrument Portfolio.
A portfolio consists of two instruments (e.g., options). Let $\mathbf{x}=(x_1,x_2)$ be a vector of positions of these two instruments, $\boldsymbol{m}=(m_1,m_2)$ be a vector of initial prices, and $\boldsymbol{y}=(y_1,y_2)$ be a vector of uncertain prices of these instruments in the next period. The loss function equals the difference between the current value of the portfolio, $(x_1m_1+x_2m_2)$, and an uncertain value of the portfolio at the next period $(x_1y_1+x_2y_2)$, i.e.,

$$f(\boldsymbol{x},\boldsymbol{y}) = (x_1m_1+x_2m_2)-(x_1y_1+x_2y_2) = x_1(m_1-y_1)+x_2(m_2-y_2)\ .$$

If we do not allow short positions, the feasible set of portfolios is a two-dimensional set of non-negative numbers

$$\boldsymbol{X} = \{(x_1,x_2),\ x_1\geq 0,\ x_2\geq 0\}\ .$$

In this case, the loss function is linear with respect to (w.r.t.) positions and the feasible set is defined by a set of linear inequalities.

For convenience, we assume that the random vector $\mathbf{y}$ has a probability density function $p(\boldsymbol{y})$. However, the existence of the density is not critical for the considered approach; this assumption can be relaxed. Denote by $\Psi(\mathbf{x},\alpha)$ the probability that the loss $f(\boldsymbol{x},\boldsymbol{y})$ does not exceed some threshold value α. The VaR function $\alpha(\boldsymbol{x},\beta)$, which is the percentile of the loss distribution with confidence level β, is the smallest number such that $\Psi(\mathbf{x},\alpha(\boldsymbol{x},\beta))=\beta$. CVaR, denoted by $\phi_\beta(\boldsymbol{x})$, which is by definition the conditional expected loss (under the condition that it exceeds VaR), is defined by

$$\phi_\beta(\mathbf{x})= (1-\beta)^{-1}\int_{f(\mathbf{x},\mathbf{y})>\alpha(\mathbf{x},\beta)} f(\mathbf{x},\mathbf{y})\,p(\mathbf{y})\,d\mathbf{y}\ .$$

It is difficult to handle CVaR because of the VaR function $\alpha(\boldsymbol{x},\beta)$ involved in its definition, unless we have an analytical representation for VaR. The main idea of our approach is that we can define a much simpler function

$$F_\beta(\mathbf{x},\alpha)= \alpha+(1-\beta)^{-1}\int_{f(\mathbf{x},\mathbf{y})>\alpha}(f(\mathbf{x},\mathbf{y})-\alpha)\,p(\mathbf{y})\,d\mathbf{y}\ ,$$

which can be used instead of CVaR. It can be proved: (1) the function $F_\beta(\mathbf{x},\alpha)$ is convex w.r.t. α; (2) VaR is a minimum point of this function w.r.t. α; and (3) that minimizing $F_\beta(\mathbf{x},\alpha)$ w.r.t. α gives CVaR

$$\phi_\beta(\mathbf{x})=F_\beta(\mathbf{x},\alpha(\mathbf{x},\beta))=\min_\alpha F_\beta(\mathbf{x},\alpha)\ .$$

This follows from the fact that the derivative of the function $F_\beta(\mathbf{x},\alpha)$ w.r.t. α equals $1+(1-\beta)^{-1}(\Psi(\mathbf{x},\alpha)-1)$, see details in [12]. By equating the derivative to zero we immediately obtain that VaR minimizes the function

$F_\beta(\mathbf{x},\alpha)$ w.r.t. α. Furthermore, we can use the function $F_\beta(\mathbf{x},\alpha)$ for the simultaneous calculation of VaR and the optimization of CVaR, i.e.,

$$\min_{\mathbf{x}\in X}\phi_\beta(\mathbf{x})=\min_{\mathbf{x}\in X,\ \alpha}F_\beta(\mathbf{x},\alpha)\ .$$

Indeed, minimization of the function $F_\beta(\mathbf{x},\alpha)$ with respect to both variables optimizes CVaR and finds VaR in "one shot". Let $(\boldsymbol{x}^*,\alpha^*)$ be a solution of the above minimization problem. Then, $F_\beta(\boldsymbol{x}^*,\alpha^*)$ equals an optimal CVaR, the optimal portfolio equals $\mathbf{x}^*$, and the corresponding VaR equals α^*. Under quite general conditions (see, [11]), the function $F_\beta(\mathbf{x},\alpha)$ is smooth. Moreover, if the function $f(\boldsymbol{x},\boldsymbol{y})$ is convex w.r.t. $\boldsymbol{x}$, then the function $F_\beta(\mathbf{x},\alpha)$ is also convex w.r.t. $\boldsymbol{x}$. Thus, if we want to minimize CVaR, we can use the convex smooth function $F_\beta(\mathbf{x},\alpha)$. Therefore, if the feasible set $\boldsymbol{X}$ is also convex, we need to solve a smooth convex optimization problem.

2. Optimization Problems with Constraints on Risk

Banks, investment companies, and other businesses tolerate different levels of risk, depending upon their objectives and capital. The adequate representation and management of risk is a critical task for business success. A typical approach in risk management is to estimate and control VaR with a specified confidence level, such as 0.95, 0.99, or 0.999. VaR is estimated for various periods, depending upon the risk management objectives - short term VaR is estimated usually for one day or two weeks, longer terms may include one, two, or five years. The problem of controlling VaR can be formalized as a mathematical programming problem with VaR constraints. However, such a problem is very difficult to solve using formal optimization methods because VaR is non-convex w.r.t. the portfolio positions and it has many local minima. In this section, we show that in contrast to VaR constraints, CVaR constraints can be easily handled using formal optimization approaches. Constraining CVaR also restricts VaR because CVaR≥VaR. Therefore, VaR constraints can be replaced by more conservative CVaR constraints.

Similar to CVaR minimization, we can include CVaR in constraints and replace it by the function $F_\beta(\mathbf{x},\alpha)$, see [8] . For instance, let us consider the problem of minimizing the mean loss $\mu(\boldsymbol{x})=Ef(\boldsymbol{x},\boldsymbol{y})$ subject to some balance constraints $\boldsymbol{x}\in\boldsymbol{X}$, and two CVaR constraints with confidence levels β and γ . In this case, the optimization problem can be stated as follows

$$\min\mu(\mathbf{x})$$

subject to

$$\boldsymbol{x}\in\boldsymbol{X},$$
$$\phi_\beta(\boldsymbol{x})\le C_\beta\ ,$$
$$\phi_\gamma(\boldsymbol{x})\le C_\gamma\ ,$$

where C_β and C_γ are some constants constraining CVaR at different confidence levels. The last two constraints can be replaced by constraints

$$F_\beta(\mathbf{x},\alpha_1)\le C_\beta\ ,$$
$$F_\gamma(\mathbf{x},\alpha_2)\le C_\gamma\ .$$

Indeed, if these constraints are satisfied for some $\alpha_1$and α_2 then they are satisfied for the minimal values $\min_{\alpha_1}F_\beta(\mathbf{x},\alpha_1)=\phi_\beta(\mathbf{x})$ and $\min_{\alpha_2}F_\gamma(\mathbf{x},\alpha_2)=\phi_\gamma(\mathbf{x})$. Optimization with these constraints assures that the CVaR values are properly restricted. Moreover, if a risk constraint is active, e.g., in the first constraint, $F_\beta(\boldsymbol{x}^*,\alpha_1^*)=C_\beta$, then the optimal value α_1^* equals β-VaR.

3. Minimizing CVaR with Finite Number of Scenarios: Linear Programming

Let us consider now the case in which an analytical representation of the density function $p(\boldsymbol{y})$ is not available, but we have J scenarios, $\boldsymbol{y}_j$, j=1,...,J, sampled from the density $p(\boldsymbol{y})$. For instance, we may have historical observations of prices for instruments of the portfolio, or we may use Monte Carlo simulations to price the instruments. In this case, the function $F_\beta(\mathbf{x},\alpha)$ can be calculated approximately as follows

$$\widetilde{F}_\beta(\mathbf{x},\alpha)=\alpha+\nu\sum_{j=1}^{J}(f(\mathbf{x},\mathbf{y}_j)-a)^+ ,$$

where the constant ν equals $\nu=((1-\beta)J)^{-1}$ and $t^+ = \max(0,t)$. If the function $f(\boldsymbol{x},\boldsymbol{y})$ is convex w.r.t. $\boldsymbol{x}$, then the function $\widetilde{F}_\beta(\mathbf{x},\alpha)$ is a convex nonsmooth function w.r.t. the vector $(\boldsymbol{x},\alpha)$. Therefore, if the feasible set $\boldsymbol{X}$ is convex, the optimization problem with the CVaR performance function can be solved using non-smooth optimization techniques. Moreover, if the function $f(\boldsymbol{x},\boldsymbol{y})$ is linear w.r.t. $\boldsymbol{x}$, these problems can be solved using LP techniques. LP approaches are routinely used in portfolio optimization with various criteria, such as mean absolute deviation [4], maximum deviation [13], and mean regret [3].

Let us first explain how LP techniques can be used for the minimization of CVaR. Indeed, after replacing in $\widetilde{F}_\beta(\mathbf{x},\alpha)$ the terms $(f(\mathbf{x},\mathbf{y}_j)-\alpha)^+$ by auxiliary varibles z_j, and imposing constraints

$$z_j \geq f(\mathbf{x},\mathbf{y}_j)-\alpha\ ,\ z_j \geq 0,$$

we can reduce minimization of the function $\widetilde{F}_\beta(\mathbf{x},\alpha)$ to the following LP problem

$$\min \quad \alpha+\nu\sum_{j=1}^{J} z_j \ , \quad (1)$$

subject to

$$\boldsymbol{x}\in \boldsymbol{X} \quad , \quad (2)$$

$$z_j \geq f(\mathbf{x},\mathbf{y}_j)-a, \quad z_j \geq 0, \quad j=1,\ldots,J. \quad (3)$$

Several case studies (see, [1, 12]) have demonstrated that this formulation provides a very powerful and numerically stable technique which can solve problems with a large number of instruments and scenarios.

Example 2. CVaR Minimization with a Constraint on Mean Losses.

Suppose that we want to minimize the CVaR of the small portfolio described in Example 1. We are interested in minimizing one day CVaR under the condition that the mean daily portfolio losses are less than or equal to $-R$ (i.e., the mean profit is bigger than or equal to R). Suppose that for two instruments in the portfolio, we have prices for J previous days. From this historical data, we can estimate J daily returns and calculate J scenarios for the next day prices, $\boldsymbol{y}_j = (y_{j1},y_{j2})$, $j=1,\ldots,J$. The mean portfolio loss equals

$$\mu(\mathbf{x})=J^{-1}\sum_{j=1}^{J} f(\mathbf{x},\mathbf{y}_j)=$$

$$J^{-1}\sum_{j=1}^{J}(x_1(m_1-y_{j1})+x_2(m_2-y_{j2})).$$

The constraint on the mean losses is formulated as follows

$$J^{-1}\sum_{j=1}^{J} (x_1(m_1-y_{j1})+x_2(m_2-y_{j2})) \leq -R \ . \quad (4)$$

The CVaR minimization problem can be easily solved by minimizing the linear function (1) subject to linear constraints (2),(3), and (4). This problem can be solved using standard LP solvers such as CPLEX.

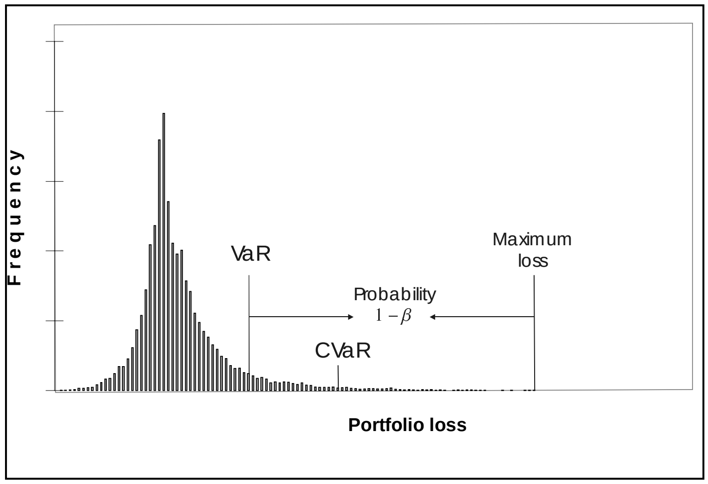

Figure 1 Portfolio Loss Distribution, VaR, and CVaR

4. Linearization of CVaR Constraints with Finite Number of Scenarios

The previous section showed that nonlinear CVaR function can be minimized using a linear objective function and linear constraints. Here, we show that a CVaR constraint in optimization problems can be approximated by a set of linear constraints. Let J scenarios, $\mathbf{y}_j$, $j=1,...,J$, be sampled from the density $p(\mathbf{y})$. Suppose that we must satisfy a CVaR constraint, $\phi_\beta(\mathbf{x}) \le C_\beta$. As it was earlier discussed, this constraint can be replaced by the constraint $F_\beta(\mathbf{x},\alpha) \le C_\beta$ using the additional variable α. Further, we can approximate this constraint by the constraint $\tilde{F}_\beta(\mathbf{x},\alpha) \le C_\beta$ using scenarios $\mathbf{y}_j$, $j=1,...,J$. Finally, the last constraint can be equivalently represented by the set of constraints

$$\alpha + \nu \sum_{j=1}^{J} z_j \le C_\beta, \qquad (5)$$

$$z_j \ge f(\mathbf{x},\mathbf{y}_j) - \alpha, \quad z_j \ge 0, \quad j=1,...,J.$$

If constraint (5) is active, then the optimal value α^* equals VaR. A case study on the application of these techniques to the optimization of the portfolio consisting of the S&P100 stocks can be found in [8].

5. Conclusion

This article has outlined a new approach for the simultaneous calculation of VaR and optimization of CVaR for a broad class of problems. We have shown that CVaR can be efficiently minimized using LP techniques. Although, formally, the method minimizes only CVaR, it also lowers VaR because CVaR ≥ VaR. Our numerical experiments show that CVaR optimal portfolios are near optimal in VaR terms, i.e., VaR can not be reduced further more than a few percent. Also, CVaR constraints can be handled efficiently using equivalent linear constraints, which dramatically improves the efficiency of the optimization techniques.

There is room for much improvement and refinement of the considered approach. For instance, the assumption that there is a joint density of instrument returns can be relaxed. LP algorithms that utilize the special structure of the minimum CVaR approach can be developed. Additional research needs to be conducted on various theoretical and numerical aspects of the methodology.

Dr. Stanislav Uryasev is a professor at the University of Florida, Dep. of Industrial and Systems Engineering, Center for Applied Optimization. His current research is concentrated on stochastic optimization algorithms and financial engineering applications, including risk management, portfolio optimization, and trading algorithms. His email and web addresses are Uryasev@ise.ufl.edu and www.ise.ufl.edu/uryasev, respectively.

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