PORTFOLIO OPTIMIZATION WITH CONDITIONAL VALUE-AT-RISK OBJECTIVE AND CONSTRAINTS

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Abstract

Recently, a new approach for optimization of Conditional Value-at-Risk (CVaR) was suggested and tested with several applications. By definition, CVaR, also called Mean Excess Loss, Mean Shortfall or Tail VaR, is the expected loss exceeding Value-at Risk (VaR). Central to the approach is an optimization technique for calculating VaR and optimizing CVaR simultaneously. This paper extends this approach to the optimization problems with CVaR constraints. In particular, the approach is used for finance applications such as maximizing returns under CVaR constraints. A case study for the portfolio of S&P 100 stocks is performed to demonstrate how the new optimization techniques can be implemented. Historical data were used for scenario generation.

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1 Introduction

Portfolio optimization has come a long way from Markowitz seminal work in 1952 [17]. Developments in portfolio optimization are stimulated by two basic requirements: (1) adequate modeling of utility functions, risks, and constraints; (2) efficiency, i.e., ability to handle large number of instruments and scenarios. These factors drove developments of such efficient tools as the mean absolute deviation approach [12], the regret optimization approach [5], and the minimax approach [34]. Mostly, high efficiency of these tools can be attributed to using linear programming techniques, rather than the Markowitz quadratic programming approach.

Recently, it was demonstrated [26] that linear programming techniques can be used for optimization of Conditional Value-at-Risk (CVaR), which is known also as Mean Excess Loss, Mean Shortfall, or Tail Value-at-Risk. A simple description of the approach for minimization of CVaR and optimization problems with CVaR constraints can be found in [33]. CVaR performance measure is closely related to Value-at-Risk (VaR). By definition, with respect to a specified probability level β, the β-VaR of a portfolio is the lowest amount α such that, with probability β, the loss will not exceed α, whereas the β-CVaR is the conditional expectation of losses above that amount α. The definition of CVaR ensures that the β-VaR is never more than the β-CVaR, so portfolios with low CVaR must have low VaR as well.

A description of various methodologies for the modeling of VaR can be seen, along with related resources, at URL http://www.gloriamundi.org/. Mostly, approaches to calculating VaR rely on linear approximation of the portfolio risks and assume a joint normal (or log-normal) distribution of the underlying market parameters [6, 9, 21, 24, 28, 31, 29]. Also, historical or Monte Carlo simulation-based tools are used when the portfolio contains nonlinear instruments such as options [4, 9, 18, 21, 24, 31, 29]. Discussions of optimization problems involving VaR can be found in [14, 15, 10, 16].

The popularity of VaR is mostly related to a simple and easy to understand representation of the high losses. Although VaR is a very popular measure of risk, it has undesirable properties [2, 3] such as lack of sub-additivity, i.e., VaR of a portfolio with two instruments may be greater than the sum of individual VaRs of these two instruments. Also, VaR is difficult to optimize when calculated using scenarios. In this case, VaR is non-convex [18] (see definition of convexity in [25]), non-smooth as a function of positions, and it has multiple local extrema. CVaR, which is a quite similar to VaR measure of risk has much more attractive properties than VaR. CVaR is more conservative than VaR and is a more consistent measure of risk since it is sub-additive

[2, 3] and convex [26]. Numerical experiments indicate that the minimization of CVaR also leads to near optimal solutions in VaR terms because CVaR is always greater than or equal to VaR. Moreover, when the return-loss distribution is normal, these two measures are equivalent [26], i.e., they provide the same optimal portfolio. Although CVaR has not become a standard in the finance industry, CVaR is gaining in the insurance industry [8]. CVaR is used also in credit risk studies [4]. Similar measures as CVaR have been earlier introduced in the stochastic programming literature, although not in financial mathematics context. The conditional expectation constraints and integrated chance constraints described in [20] may serve the same purpose as CVaR.

Several case studies showed that risk optimization with the CVaR performance function and constraints can be done for large portfolios and a large number of scenarios with relatively small computational resources. A case study on the hedging of a portfolio of options using the CVaR minimization technique is included in [26]. This problem was first studied at Algorithmics, Inc. with the minimum expected regret approach [18]. Also, the CVaR minimization approach was applied to credit risk management of a portfolio of bonds [1]. This portfolio was put together by several banks to test various credit risk modeling techniques. Earlier, the minimum expected regret optimization technique was applied to the same portfolio at Algorithmics, Inc. [19]. Paper [1] have used the same set of scenarios to test the minimum CVaR technique. The reader interested in other applications of optimization techniques in the finance area can find relevant papers in [35].

This paper extends the CVaR minimization approach [26] for optimizing portfolios of financial instruments. We show that the approach [26] can be used also for maximizing returns under a CVaR constraint as opposed to minimizing CVaR. First, we briefly describe the approach from [26] to lay foundation for the further extension of this approach. Second, we formulate a general theorem on various equivalent representations of the efficient frontier with convex risk performance functions. This equivalence is well known for mean-variance, see for instance, [30], and for mean-regret [5] performance functions. We have shown that it holds for any convex risk function, in particular for the CVaR function considered in this paper. Third, using an auxiliary variable, we formulated a theorem on reduction of the problem with CVaR constraints to a much simpler convex problem. Similar result is formulated also for the case when both return and CVaR are included in the performance function. As it was earlier identified in [26], optimization automatically sets the auxiliary variable to VaR which significantly simplifies the problem solution (recall that CVaR is defined as a mean losses exceeding VaR). Further, when the distribution is given by fixed number of scenarios and the loss function is linear, we showed how the CVaR function can be replaced by a linear function and an additional set of linear constraints. Fourth, to test the approach, we developed a one-period model for optimization of a portfolio of stocks using historical scenario generation. A case study was conducted on the optimization of S&P100 portfolio of stocks with CVaR constraints. Finally, formal proofs of theorems are included in the appendix after the list of references.

2 Conditional Value-at-Risk

Approach developed in [26] provides the foundation for the analysis conducted in this paper. First, following [26], we formally define CVaR and present theorems which are needed for understanding of this paper. Let $f(\mathbf{x}, \mathbf{y})$ be the loss associated with the decision vector $\mathbf{x}$, to be chosen from a certain subset $\mathbf{X}$ of $\mathbb{R}^n$, and the random vector $\mathbf{y}$ in $\mathbb{R}^m$. (We use boldface type for vectors to distinguish them from scalars.) The vector $\mathbf{x}$ can be interpreted as representing a portfolio, with $\mathbf{X}$ as the set of available portfolios (subject to various constraints), but other interpretations could be made as well. The vector $\mathbf{y}$ stands for the uncertainties, e.g., some market parameters, that can affect the loss. Of course the loss might be negative and thus, in effect, constitute a gain.

For each $\mathbf{x}$, the loss $f(\mathbf{x}, \mathbf{y})$ is a random variable having a distribution in $\mathbb{R}$ induced by that of $\mathbf{y}$. The underlying probability distribution of $\mathbf{y}$ in $\mathbb{R}^m$ will be assumed for convenience to have density, which we denote by $p(\mathbf{y})$. This assumption is not critical for the considered approach and can be relaxed. The probability of $f(\mathbf{x}, \mathbf{y})$ not exceeding a threshold α is then given by

$$\Psi(\mathbf{x}, \alpha) \quad = \quad \int_{f(\mathbf{x}, \mathbf{y}) \leq \alpha} p(\mathbf{y})\, d\mathbf{y}. \tag{1}$$

As a function of α for fixed $\mathbf{x}$, $\Psi(\mathbf{x}, \alpha)$ is the cumulative distribution function for the loss associated with $\mathbf{x}$. It completely determines the behavior of this random variable and is fundamental in defining VaR and CVaR.

The function $\Psi(\mathbf{x}, \alpha)$ is nondecreasing with respect to (w.r.t.) α and we assume that $\Psi(\mathbf{x}, \alpha)$ is everywhere continuous w.r.t. α. This assumption, like the previous one about density in $\mathbf{y}$, is made for simplicity. In some common situations, the required continuity follows from properties of the loss $f(\mathbf{x}, \mathbf{y})$ and the density $p(\mathbf{y})$; see [32].

The β-VaR and β-CVaR values for the loss random variable associated with $\mathbf{x}$ and any specified probability level β in $(0, 1)$ will be denoted by $\alpha_{\beta(\mathbf{x})}$ and $\phi_{\beta(\mathbf{x})}$. In our setting they are given by

$$\alpha_\beta(\mathbf{x}) \;=\; \min\{\, \alpha \in I\!R : \; \Psi(\mathbf{x},\alpha) \ge \beta \,\} \tag{2}$$

and

$$\phi_\beta(\mathbf{x}) \;=\; (1-\beta)^{-1} \int_{f(\mathbf{x},\mathbf{y}) \ge \alpha_\beta(\mathbf{x})} f(\mathbf{x},\mathbf{y}) p(\mathbf{y})\, d\mathbf{y}. \tag{3}$$

In the first formula, $\alpha_\beta(\mathbf{x})$ comes out as the left endpoint of the nonempty interval consisting of the values α such that actually $\Psi(\mathbf{x},\alpha) = \beta$. (This follows from $\Psi(\mathbf{x},\alpha)$ being continuous and nondecreasing w.r.t. α. The interval might contain more than a single point if Ψ has "flat spots.") In the second formula, the probability that $f(\mathbf{x},\mathbf{y}) \ge \alpha_\beta(\mathbf{x})$ is therefore equal to $1-\beta$. Thus, $\phi_\beta(\mathbf{x})$ comes out as the conditional expectation of the loss associated with $\mathbf{x}$ relative to that loss being $\alpha_\beta(\mathbf{x})$ or greater.

The key to the approach is a characterization of $\phi_\beta(\mathbf{x})$ and $\alpha_\beta(\mathbf{x})$ in terms of the function F_β on $X \times I\!R$ that we now define by

$$F_\beta(\mathbf{x},\alpha) \;=\; \alpha + (1-\beta)^{-1} \int_{y \in \boldsymbol{R}^n} [f(\mathbf{x},\mathbf{y}) - \alpha]^+ \, p(\mathbf{y})\, d\mathbf{y}, \tag{4}$$

where $[t]^+ = t$ when $t > 0$ but $[t]^+ = 0$ when $t \le 0$. The crucial features of F_β, under the assumptions made above, are as follows.

The following theorem is proved in [26].

Theorem 1. *As a function of α, $F_\beta(\mathbf{x},\alpha)$ is convex and continuously differentiable. The β-CVaR of the loss associated with any $\mathbf{x} \in X$ can be determined from the formula*

$$\phi_\beta(\mathbf{x}) \;=\; \min_{\alpha \in \boldsymbol{R}} F_\beta(\mathbf{x},\alpha). \tag{5}$$

In this formula, the set consisting of the values of α for which the minimum is attained, namely

$$A_\beta(\mathbf{x}) \;=\; \operatorname*{argmin}_{\alpha \in \boldsymbol{R}} F_\beta(\mathbf{x},\alpha), \tag{6}$$

is a nonempty, closed, bounded interval (perhaps reducing to a single point), and the β-VaR of the loss is given by

$$\alpha_\beta(\mathbf{x}) \;=\; \textit{left endpoint of } A_\beta(\mathbf{x}). \tag{7}$$

In particular, one always has

$$\alpha_\beta(\mathbf{x}) \in \operatorname*{argmin}_{\alpha \in \boldsymbol{R}} F_\beta(\mathbf{x},\alpha) \quad \textit{and} \quad \phi_\beta(\mathbf{x}) = F_\beta(\mathbf{x}, \alpha_\beta(\mathbf{x})). \tag{8}$$

For background on convexity, which is a key property in optimization that in particular eliminates the possibility of a local minimum being different from a global minimum, see [25, 27], for instance. Other important advantages of viewing VaR and CVaR through the formulas in Theorem 1 are captured in the next theorem, also proved in [26].

Theorem 2. *Minimizing the β-CVaR of the loss associated with $\mathbf{x}$ over all $\mathbf{x} \in \mathbf{X}$ is equivalent to minimizing $F_\beta(\mathbf{x}, \alpha)$ over all $(\mathbf{x}, \alpha) \in \mathbf{X} \times I\!R$, in the sense that*

$$\min_{\mathbf{x}\in\mathbf{X}} \phi_\beta(\mathbf{x}) \;=\; \min_{(\mathbf{x},\alpha)\in X\times \boldsymbol{R}} F_\beta(\mathbf{x}, \alpha), \tag{9}$$

where moreover a pair $(\mathbf{x}^*, \alpha^*)$ achieves the right hand side minimum if and only if $\mathbf{x}^*$ achieves the left hand side minimum and $\alpha^* \in A_\beta(\mathbf{x}^*)$. In particular, therefore, in circumstances where the interval $A_\beta(\mathbf{x}^*)$ reduces to a single point (as is typical), the minimization of $F(\mathbf{x}, \alpha)$ over $(\mathbf{x}, \alpha) \in \mathbf{X} \times I\!R$ produces a pair $(\mathbf{x}^*, \alpha^*)$, not necessarily unique, such that $\mathbf{x}^*$ minimizes the β-CVaR and α^* gives the corresponding β-VaR.

Furthermore, $F_\beta(\mathbf{x}, \alpha)$ is convex w.r.t. $(\mathbf{x}, \alpha)$,and $\phi_\beta(\mathbf{x})$ is convex w.r.t. $\mathbf{x}$, when $f(\mathbf{x}, \mathbf{y})$ is convex with respect to $\mathbf{x}$, in which case, if the constraints are such that $\mathbf{X}$ is a convex set, the joint minimization is an instance of convex programming.

According to Theorem 2, it is not necessary, for the purpose of determining a $\mathbf{x}$ that yields the minimum β-CVaR, to work directly with the function $\phi_\beta(\mathbf{x})$, which may be hard to do because of the nature of its definition in terms of the β-VaR value $\alpha_\beta(\mathbf{x})$ and the often troublesome mathematical properties of that value. Instead, one can operate on the far simpler expression $F_\beta(\mathbf{x}, \alpha)$ with its convexity in the variable α and even, very commonly, with respect to $(\mathbf{x}, \alpha)$.

3 Efficient Frontier: Different Formulations

Paper [26] considered minimization of CVaR while requiring a minimum expected return. However, usually it is needed to maximize returns while not allowing large risks. We, therefore, swapped the CVaR function and the expected return in the problem formulation (compared to [26]), thus minimizing the negative of the expected return with a CVaR constraint. We will show on a general level that there are three equivalent formulations of the optimization problem. They are equivalent in the sense that they produce the same efficient frontier.

The efficient frontier is made up of points such that given the return and risk of a point on the frontier, one can not achieve the same or higher return with a lower risk, nor the same or lower risk with a higher return.

Theorem 3. *Consider a risk function $\phi_\beta(\mathbf{x})$ and a reward function $R(\mathbf{x})$, both dependent on the decision vector $\mathbf{x}$, and consider the following three problems:*

$$\text{(P1)} \qquad \min_{x} \quad \phi_\beta(\mathbf{x}) - \mu_1 R(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X}, \quad \mu_1 \geq 0,$$

$$\text{(P2)} \qquad \min_{x} \quad \phi_\beta(\mathbf{x}), \quad R(\mathbf{x}) \geq \rho, \quad \mathbf{x} \in \mathbf{X},$$

$$\text{(P3)} \qquad \min_{x} \quad -R(\mathbf{x}), \quad \phi_\beta(\mathbf{x}) \leq \omega, \quad \mathbf{x} \in \mathbf{X}.$$

Varying the parameters μ_1, ρ, and ω, traces the efficient risk-return frontiers for the problems (P1)-(P3), accordingly. If $\phi_\beta(\mathbf{x})$ is convex, $R(\mathbf{x})$ is concave and the set $\mathbf{X}$ is convex, then the three problems, (P1)-(P3), generate the same efficient frontier.

The proof of Theorem 3 is furnished in Appendix A. From Theorem 2 we have that $\phi_\beta(\mathbf{x})$ is convex if the loss function $f(\mathbf{x}, \mathbf{y})$ is convex w.r.t. $\mathbf{x}$. Since, further in this paper, both the return function, $R(\mathbf{x})$, and the constraint matrix are linear, the conditions of Theorem 3 are satisfied, In this paper, we always assume that $\phi_\beta(\mathbf{x})$ is convex, $R(\mathbf{x})$ is concave and the set $\mathbf{X}$ is convex. Therefore, maximizing return under a CVaR constraint, generates the same efficient frontier as the minimization of CVaR performed in [26].

The equivalence between problems (P1)-(P3) is well known for mean-variance [30] and mean-regret [5] performance functions. We have shown that it holds for any convex risk function, in particular for the CVaR function considered in this paper.

4 Equivalent Formulations with Auxiliary Variables

Theorem 3 implies that we can use problem formulation (P3) or (P1) instead of (P2) for generating the efficient frontier. Theorem 2 shows that the function $F_\beta(\mathbf{x}, \alpha)$ can be used instead of $\phi_\beta(\mathbf{x})$ to solve problem (P2). Further, we demonstrate that, similar, the function $F_\beta(\mathbf{x}, \alpha)$ can be used instead of $\phi_\beta(\mathbf{x})$ in problems (P1) and (P3).

Theorem 4. *The two minimization problems below*

$$\text{(P3)} \qquad \min_{\mathbf{x} \in \mathbf{X}} \quad -R(\mathbf{x}), \quad \phi_\beta(\mathbf{x}) \leq \omega, \quad \mathbf{x} \in \mathbf{X}$$

and

$$\text{(P3}'\text{)} \qquad \min_{(\alpha, \mathbf{x}) \in \mathbf{X} \times I\!R} \quad -R(\mathbf{x}), \quad F_\beta(\mathbf{x}, \alpha) \leq \omega, \quad \mathbf{x} \in \mathbf{X}$$

are equivalent in the sense that their objectives achieve the same minimum values. Moreover, if the CVaR constraint in (P3) is active, a pair $(\mathbf{x}^*, \alpha^*)$ *achieves the minimum of (P3′) if and only if* $\mathbf{x}^*$ *achieves the minimum of (P3) and* $\alpha^* \in A_\beta(\mathbf{x}^*)$*. In particular, when the interval* $A_\beta(\mathbf{x}^*)$ *reduces to a single point, the minimization of* $-R(\mathbf{x})$ *over* $(\mathbf{x}, \alpha) \in \mathbf{X} \times I\!R$ *produces a pair* $(\mathbf{x}^*, \alpha^*)$ *such that* $\mathbf{x}^*$ *maximizes the return and* α^* *gives the corresponding* β*-VaR.*

Theorem 5. *The two minimization problems below*

$$\text{(P1)} \qquad \min_{\mathbf{x} \in \mathbf{X}} \quad \phi_\beta(\mathbf{x}) - \mu_1 R(\mathbf{x}), \quad \mu_1 \geq 0, \quad \mathbf{x} \in \mathbf{X}$$

and

$$\text{(P1}'\text{)} \qquad \min_{(\alpha, \mathbf{x}) \in \mathbf{X} \times I\!R} \quad F_\beta(\mathbf{x}, \alpha) - \mu_1 R(\mathbf{x}), \quad \mu_1 \geq 0, \quad \mathbf{x} \in \mathbf{X}$$

are equivalent in the sense that their objectives achieve the same minimum values. Moreover a pair $(\mathbf{x}^*, \alpha^*)$ *achieves the minimum of (P1′) if and only if* $\mathbf{x}^*$ *achieves the minimum of (P1) and* $\alpha^* \in A_\beta(\mathbf{x}^*)$*. In particular, when the interval* $A_\beta(\mathbf{x}^*)$ *reduces to a single point, the minimization of* $F_\beta(\mathbf{x}, \alpha) - \mu_1 R(\mathbf{x})$ *over* $(\mathbf{x}, \alpha) \in \mathbf{X} \times I\!R$ *produces a pair* $(\mathbf{x}^*, \alpha^*)$ *such that* $\mathbf{x}^*$ *minimizes* $\phi_\beta(\mathbf{x}) - \mu_1 R(\mathbf{x})$ *and* α^* *gives the corresponding* β*-VaR.*

The proof of Theorems 4 and 5 are furnished in Appendix B.

5 Discretization

The equivalent problem formulations presented in Theorems 2, 4 and 5 can be combined with ideas for approximating the integral in $F_\beta(\mathbf{x}, \alpha)$, see (4). This offers a rich range of possibilities.

The integral in $F_\beta(\mathbf{x}, \alpha)$ can be approximated in various ways. For example, this can be done by sampling the probability distribution of $\mathbf{y}$ according to its density $p(\mathbf{y})$. If the sampling generates a collection of vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_J$, then the corresponding approximation to

$$F_\beta(\mathbf{x}, \alpha) = \alpha + (1 - \beta)^{-1} \int_{y \in \boldsymbol{R}^n} (f(\mathbf{x}, \mathbf{y}) - \alpha)^+ p(\mathbf{y}) d\mathbf{y}$$

is

$$\tilde{F}_\beta(\mathbf{x}, \alpha) \; = \; \alpha + \nu \sum_{j=1}^{J} [f(\mathbf{x}, \mathbf{y}_j) - \alpha]^+ \,, \tag{10}$$

where $\nu = J^{-1}(1 - \beta)^{-1}$. If the loss function $f(\mathbf{x}, \mathbf{y})$ is linear w.r.t. $\mathbf{x}$, then the function $\tilde{F}_\beta(\mathbf{x}, \alpha)$ is convex and piecewise linear.

6 Linearization

The function $F_\beta(\mathbf{x},\alpha)$ in optimization problems in Theorems 2,4, and 5 can be approximated by the function $\tilde{F}_\beta(\mathbf{x},\alpha)$. Further, using dummy variables z_j, $j = 1, ..., J$, the function $\tilde{F}_\beta(\mathbf{x},\alpha)$ can be replaced by the linear function $\alpha + \nu \sum_{j=1}^{J} z_j$ and the set of linear constraints

$$z_j \geq f(\mathbf{x}, \mathbf{y}_j) - \alpha, \qquad z_j \geq 0, \qquad j = 1, ..., J.$$

For instance, using Theorem 4 we can replace the constraint

$$\phi_\beta(\mathbf{x}) \leq \omega$$

in optimization problem (P3) by the constraint

$$F_\beta(\mathbf{x},\alpha) \leq \omega \ .$$

Further, the above constraint can be approximated by

$$\tilde{F}_\beta(\mathbf{x},\alpha) \leq \omega \ , \tag{11}$$

and reduced to the following system of linear constraints

$$\alpha + \nu \sum_{j=1}^{J} z_j \leq \omega, \tag{12}$$

$$z_j \geq f(\mathbf{x}, \mathbf{y}_j) - \alpha, \qquad z_j \geq 0, \qquad j = 1, ..., J. \tag{13}$$

Similar, approximations by linear functions can be done in optimization problems in Theorems 2 and 5.

7 One Period Portfolio Optimization Model with Transaction Costs

Let us consider a portfolio of n different financial instruments in the market, s_i, $(i = 1, ..., n)$. Let $\mathbf{x}^0 = (x_1^0, x_2^0, ..., x_n^0)$ be the positions, e.g., number of shares, of each instrument in the initial portfolio, and let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be the positions in the optimal portfolio that we intend to find using the algorithm. Both, $\mathbf{x}^0$ and $\mathbf{x}$, describe how many of each instrument there are in the respective portfolios. The initial prices for the instruments are given by $\mathbf{q} = (q_1, q_2, ..., q_n)$. The product $\mathbf{q}^T\mathbf{x}^0$ is thus the initial portfolio value. The scenario-dependent prices for each instrument at the end of the period are given by $\mathbf{y} = (y_1, y_2, ..., y_n)$. The transaction cost, c_i, is defined as a fixed percentage cost of the value traded. The loss function over the period is

$$f(\mathbf{x}, \mathbf{y}, \mathbf{x}^0, \mathbf{q}) = -\mathbf{y}^T\mathbf{x} + \mathbf{q}^T\mathbf{x}^0. \tag{14}$$

Further, we will consider the optimization problem (P3′) with the function $F_\beta(\mathbf{x}, \alpha)$ discretized according to (10) and linearized with the set of constraints (12) and (13).

7.1 Return Performance Function

The objective function is defined as the return over the optimization period, which is the expected value at the end of the period divided by the initial value. Let us denote by v, the initial value of the portfolio, $v = \sum_{i=1}^n q_i x_i^0$. Then, the objective function to be minimized, equals

$$R(\mathbf{x}) = \frac{1}{v} \sum_{i=1}^{n} -\mathbb{E}[y_i] x_i \ . \tag{15}$$

7.2 Cash

Among the n instruments available, we define one as cash. This allows the model to be used without modifications when an investor starts out with cash in the first period. Most investors also hold some cash in the following periods. It generates a low but certain return, r_{cash}.

7.3 Transaction costs

In this paper, we assume a linear transaction cost, proportional to the value of the shares traded. For a treatment of non-convex transaction costs, see [11]. With every instrument, we associate a transaction cost c_i. When buying or selling instrument i, one pays c_i times the value of the stocks traded. For cash we set $c_{cash} = 0$. That is, one only pays for buying and selling the instrument, and not for moving the cash in and out of the account.

First, we consider the balance constraint that maintains the total value of the portfolio less transaction costs

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i q_i |x_i^0 - x_i| + \sum_{i=1}^{n} q_i x_i \ .$$

Since we assume a proportional trading fee, the problem can also be formulated using the following set of linear constraints

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i q_i (\underline{\delta}_i + \overline{\delta}_i) + \sum_{i=1}^{n} q_i x_i \ ,$$

$$x_i^0 - \underline{\delta}_i + \overline{\delta}_i = x_i, \qquad i = 1, ..., n,$$

$$\underline{\delta}_i \geq 0,\ \overline{\delta}_i \geq 0, \qquad i = 1, ..., n.$$

The nonlinear constraint $\underline{\delta}_i \overline{\delta}_i = 0$ can be omitted since simultaneous buying and selling of the same instrument, i, can never be optimal.

As is apparent from the formulation of the conditions above, we could just as well have used a transaction cost proportional to the number of shares traded. In the value maintaining condition, we would have c_i instead of $c_i q_i$.

7.4 Change in Individual Positions

We consider that the position changes can be bounded. This bound could be, for example, a fixed number or be proportional to the initial position in the instrument

$$0 \leq \underline{\delta}_i \leq \underline{\delta}_i^{max}, \quad 0 \leq \overline{\delta}_i \leq \overline{\delta}_i^{max}, \qquad i = 1, ..., n.$$

7.5 Bounds on Positions

We, also, consider that the positions themselves can be bounded

$$\underline{x}_i \leq x_i \leq \overline{x}_i, \qquad i = 1, ..., n. \tag{16}$$

7.6 Value Constraint

We do not allow for an instrument i to constitute more than a given percent, ν_i, of the total portfolio value

$$q_i x_i \leq \nu_i \sum_{i=1}^{n} x_i q_i \ .$$

This constraint of course has sense only when short positions are not allowed.

7.7 CVaR Constraint

Current regulations impose capital requirements for investment companies, proportional to the VaR of a portfolio. This requirement can be enforced by the constraint on CVaR since CVaR $\geq$ VaR. The upper bound on CVaR, ω, could be chosen as the maximum VaR. Here we use the linearized set of constraints

$$\alpha + \nu \sum_{j=1}^{J} z_j \leq \omega \ , \tag{17}$$

$$z_j \geq \sum_{i=1}^{n} (-y_{ij} x_i + q_i x_i^0) - \alpha \ , \qquad z_j \geq 0, \qquad j = 1, ..., J. \tag{18}$$

7.8 Optimization Problem

Below we present the optimization problem described in sections (7.1)-(7.7).

$$\min_{\mathbf{x},\alpha} \frac{1}{v} \sum_{i=1}^{n} -\mathbb{E}[y_i]x_i, \tag{19}$$

subject to the constraints

$$\alpha + \nu \sum_{j=1}^{J} z_j \leq \omega, \tag{20}$$

$$z_j \geq \sum_{i=1}^{n}(-y_{ij}x_i + q_i x_i^0) - \alpha, \qquad z_j \geq 0, \qquad j = 1, ..., J, \tag{21}$$

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i(\underline{\delta}_i + \overline{\delta}_i) + \sum_{i=1}^{n} q_i x_i, \tag{22}$$

$$x_i^0 - \underline{\delta}_i + \overline{\delta}_i = x_i, \qquad i = 1, ..., n, \tag{23}$$

$$0 \leq \underline{\delta}_i \leq \underline{\delta}_i^{max}, \quad 0 \leq \overline{\delta}_i \leq \overline{\delta}_i^{max}, \qquad i = 1, ..., n, \tag{24}$$

$$\underline{x}_i \leq x_i \leq \overline{x}_i, \qquad i = 1, ..., n, \tag{25}$$

$$x_i \leq \nu_i \sum_{k=1}^{n} x_k q_k, \qquad i = 1, ..., n. \tag{26}$$

By solving problem (19)-(26) above, we get the optimal vector $\mathbf{x}^*$, the corresponding VaR, which equals α^*, and the maximum expected return, which equals $\frac{1}{v}\mathbb{E}[\mathbf{y}]\mathbf{x}^*$. By solving problem (19)-(26) for different portfolio CVaRs, ω, we get the efficient return-CVaR frontier for the portfolio.

7.9 Size of LP

For n instruments and J scenarios, the formulation of the LP problem presented above has $3n + J + 1$ variables and $2(n + 1) + J$ constraints, not including bounds on $\underline{\delta}_i$, $\overline{\delta}_i$ and x_i. The number of nonzero coefficients in the constraint matrix is $6n + nJ + n^2 + 3J + 1$.

7.10 Scenario Generation

With our approach, the integral in the CVaR function is approximated by the sum over all scenarios. The approach can be used with different schemes for generating scenarios [23]. One can assume a joint distribution for the price-return process for all instruments and generate scenarios in a Monte Carlo simulation. Also, the approach allows for easy use of historical data without assuming a particular distribution. When generating the scenarios, we start with historic time series of prices for the n instruments. Dividing these series into J periods (scenarios), we can calculate the return over each of these periods. The length of the period, Δt, is the same as the length of the period over which we optimize the portfolio. For instance, minimizing over a one day period, we take the closing prices of two consecutive days, p^t and p^{t+1}. Similarly, for a two week period, we start with the closing price at trading day 1 and take the closing price at trading day 10 and then we step forward to day 2 and take the prices at days 2 and 11, and so on. For every pair of consecutive historic prices, we generate a scenario with the corresponding return, for every instrument i,

$$r_{ij}^h = p_i^{t+\Delta t}/p_i^t \ .$$

The h indicates that the numbers are generated using historic data. From these scenarios, we can calculate the expected return for each instrument

$$\mathbb{E}[r_i^h] = J^{-1} \sum_{j=1}^{J} r_{ij}^h \ .$$

These returns can be used to calculate prices for each scenario and the expected price for each instrument at the end of the period over which we optimize,

$$y_{ij}^h = q_i r_{ij}^h,$$

$$\mathbb{E}[y_i^h] = J^{-1} \sum_{j=1}^{J} y_{ij}^h.$$

Further, in the numerical simulations, we consider a two week period.

8 Case Study: Portfolio of S&P 100 Stocks

We now proceed with a case study and construct the efficient frontier of a portfolio consisting of stocks in the S&P100 index. We maximized the portfolio value subject to various constraints on CVaR. The algorithm was implemented in C++ and we used CPLEX 4.0 Callable Library to solve the LP problem. The optimization was run for the two-week period, ten business days, from June 14, 1999 to June 28, 1999. The initial portfolio contained only cash in amount of 10,000 U.S. dollars. For scenario generation we used the five hundred most recent overlapping two-week periods. That is, the oldest data we used were from June 1, 1997. The CVaR constraint, ω was set equal to the percentage of the initial portfolio value. The limits on the positions were set to $\underline{x}_i = 0$ and $\overline{x}_i = \infty$ respectively. We do not allow short positions. The limits on the changes in the individual positions, $\underline{\delta}^{max}$ and $\overline{\delta}^{max}$, were both set to infinity. The limit on how large a part of the total portfolio value one single asset can constitute, ν_i, was set to 20 percent for all i. The return on any cash was set to 0.16 percent over two weeks.

8.1 Set of Instruments

As stated above, the set of instruments to invest in was set to the stocks in the S&P100 as of the first of September 1999. Due to insufficient data, six of the stocks were excluded[1]. We used historical price data (end-of-the-day prices) from 1997-07-14 to 1999-07-28, and the investment date was 1999-07-14.

Fig. 1 shows the efficient frontiers for the portfolio under CVaR optimization. Rate of return was calculated as the ratio of the optimized portfolio value to the initial value; risk is the percentage of the initial portfolio value which is allowed to be risked. For example, setting risk $= 0.10$ and $\beta = 0.95$ means that we allow for a 10% loss of the initial portfolio value with a probability of 5%. It is apparent from the figure that larger β-CVaR constraint values allow us to achieve higher returns. In this numerical example, the maximum rate of return of the optimized portfolio equals 1.29 .

Table 1 presents the portfolio configurations for different risk levels ($\beta = 0.90$). Earlier, we imposed constraint on the percentage of the total portfolio value that one stock can constitute (26). In the numerical experiment, we set ν equal to 0.2, so a single asset cannot constitute more than 20% of the total portfolio value. Analyzing the data in Table 1, we see that for higher levels of allowed risk, the algorithm reduces the number of the instruments in the portfolio in order to achieve a higher return (due to the imposed constraints, the minimal number of stocks in the portfolio equals five). This confirms the well-known fact that 'diversifying' the portfolio reduces the risk. 'Loosening' the constraint on risk causes the algorithm to choose only a few of the 'most profitable' stocks. As we decrease our level of risk, the number of instruments in the portfolio

[1]Citigroup Inc., Hartford Financial Svc.Gp., Lucent Technologies, Mallinckrodt Inc., Raytheon Co., U.S. Bancorp.

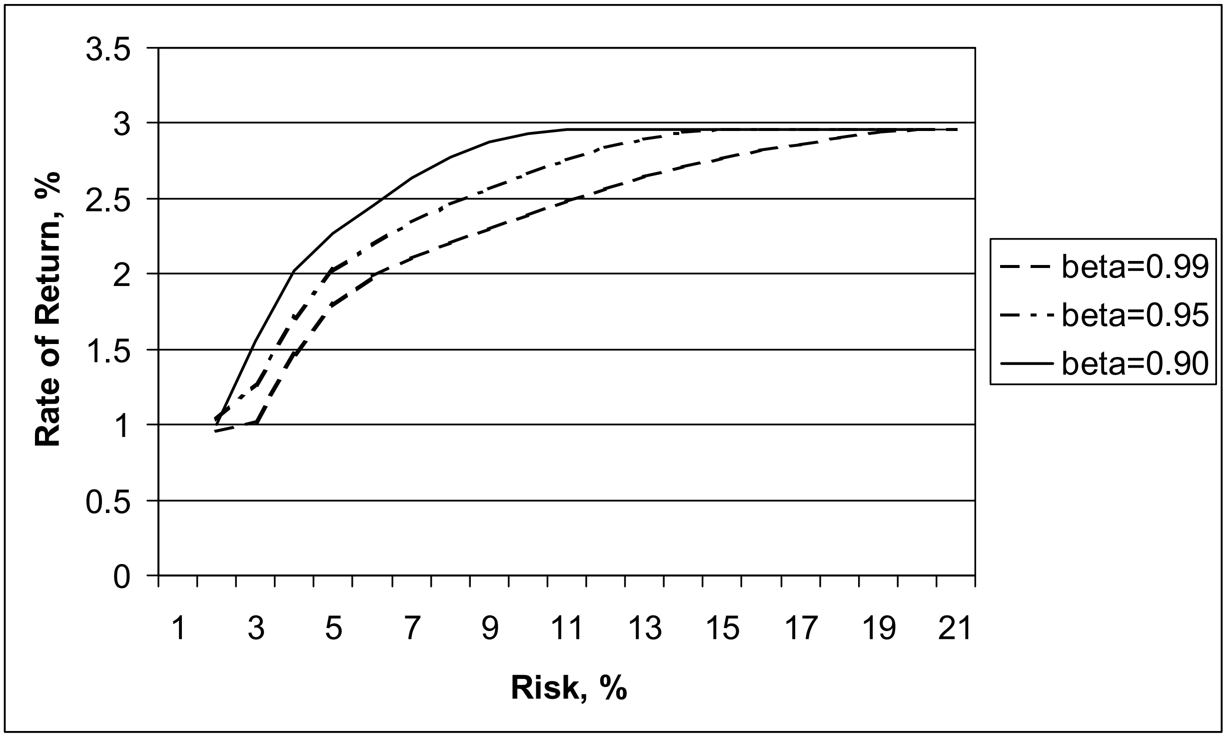

Figure 1: Efficient frontier (optimization with CVaR constraints). Rate of Return is the rate of return of the optimized portfolio during a 2 week period. Risk is the percentage o the portfolio value, which is allowed to be lost with probability 1-β during 2 week period.

increases, and for 'conservative' investing (1 or 2% of risk), we obtain a portfolio with more than 10 assets, which also includes cash (the least risky asset). For 1%-risk the share of cash in the portfolio will be greater if we relax the constraint (26) for the cash, i.e. the algorithm would invest more in the 'risk-free' cash.

8.2 Transaction Cost

Transaction costs need to be taken into account when employing an active trading strategy. Except fee paid to the broker/market, there are costs induced by bid-ask spreads and poor liquidity. To examine impact of the transaction costs, we calculated the efficient frontier with the following transaction costs, $c = 0\%, 1\%$, and 4%. From Figure 2, we can see how much the transaction cost lowers the return. The transaction cost does not lower the return linearly. Since it is incorporated into the optimization, it can also affect the choice of stocks.

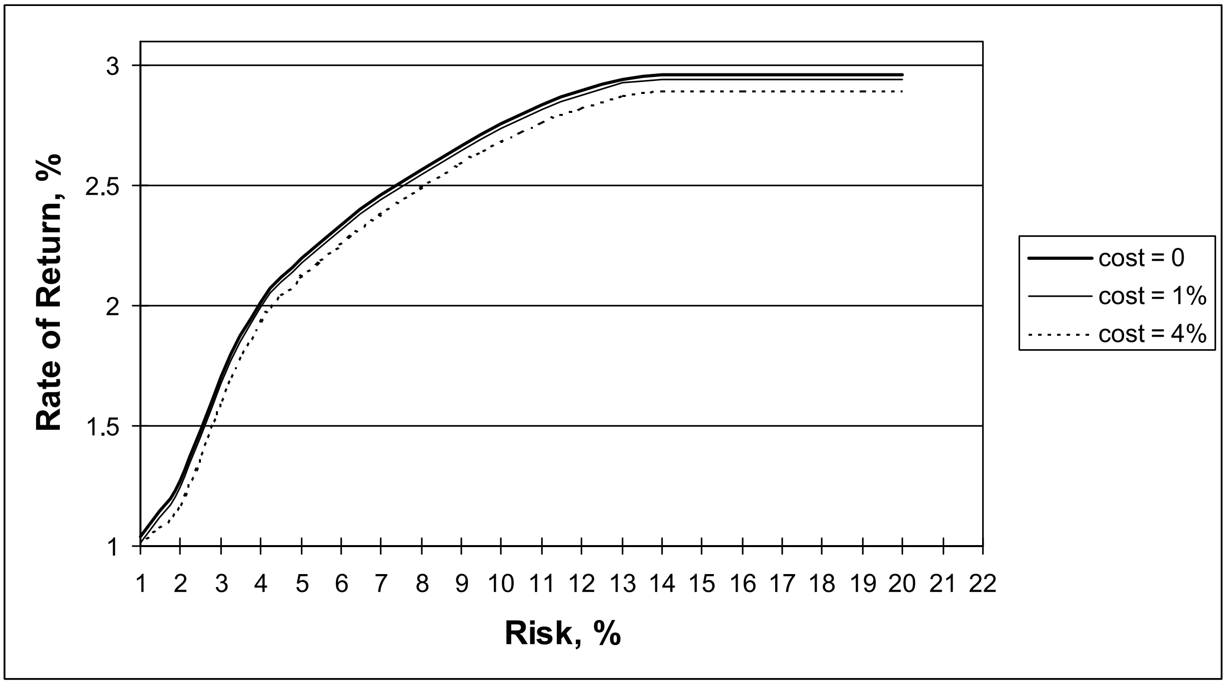

Figure 2: Efficient frontier with transaction costs $c = 0\%, 1\%$, and 4% (optimization with 0.95-CVaR constraints). Rate of Return is the rate of return of the optimized portfolio during 2 week period. Risk is the percentage of the portfolio value, which is allowed to be lost with a probability of 0.05 during a 2 week period.

9 Concluding Remarks

The paper extends the approach for portfolio optimization [26] which simultaneously calculates VaR and optimizes CVaR. We first showed (Theorem 3) that for risk-return optimization problems with convex constraints, one can use different optimization formulations. This is true in particular for the considered CVaR optimization problem. We then showed (Theorems 4 and 5) that the approach [26] can be extended to the reformulated problems with CVaR constraints and the weighted return-CVaR performance function. We developed a model for optimizing portfolio returns with CVaR constraints using historical scenarios. We conducted a case study on optimizing the portfolio of S&P100 stocks with return performance function and CVaR constraints. The case study showed that the optimization algorithm, which is based on linear programming techniques, is very stable and efficient. Numerical experiments demonstrated that the suggested portfolio optimization approach with CVaR constraints may lead to new efficient investment strategies.

Table 1: Portfolio configuration: number of shares of stock in the optimized portfolio depending on the risk level.

Risk	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
Cash	2000	960.16	0	0	0	0	0	0	0	0
AA	1.25	1.30	0	0	0	0	0	0	0	0
AIT	0	11.90	24.09	24.61	28.32	28.32	17.12	0	0	0
AVP	0	0	0.63	0	0	0	0	0	0	0
BAX	4.42	0	0	0	0	0	0	0	0	0
BEL	3.25	2.28	3.63	0	0	0	0	0	0	0
CSC	0	0.67	0	0	0	0	0	0	0	0
CSCO	0	0	0	3.55	7.97	21.96	30.62	30.62	30.62	30.62
ETR	33.99	10.54	0	0	0	0	0	0	0	0
GD	10.58	13.26	8.45	0	0	0	0	0	0	0
HM	50.21	0	0	0	0	0	0	0	0	0
IBM	5.43	10.44	11.12	5.51	3.32	0	1.80	1.39	0	9.60
IFF	1.59	0	0	0	0	0	0	0	0	0
LTD	5.48	5.63	3.59	0	0	0	0	0	0	0
MOB	4.08	5.57	0	0	0	0	0	0	0	0
MSFT	0	0	0	0	0	0	0	0	0	7.19
MTC	0	0	0	0	0	0	0	0	0	0
SO	26.45	11.02	0	0	0	0	0	0	0	0
T	11.46	21.08	31.66	35.52	35.52	35.52	35.52	35.52	17.03	0
TAN	8.98	18.33	28.03	32.50	37.78	37.78	37.78	37.78	37.78	37.78
TXN	0	0	0	0	0	0	0	12.42	26.08	26.73
UCM	47.62	47.62	47.62	47.62	29.92	13.47	0	0	0	0
UIS	0	0	9.66	27.79	40.11	45.39	45.39	45.39	45.39	45.39
WMT	0	0	1.56	6.90	0	0	11.42	18.47	22.89	0

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Appendix A. Proof of Theorem 3

The proof of Theorem 3 is based on the Kuhn-Tucker necessary and sufficient conditions stated in the following theorem.

Theorem A1 (Kuhn-Tacker, Theorem 2.5, [22]). *Consider the problem*

$$\min \psi_0(\mathbf{x}),$$

$$\psi_i(\mathbf{x}) \leq 0 \qquad i = -m, ..., -1,$$

$$\psi_i(\mathbf{x}) = 0 \qquad i = 1, ..., n,$$

$$\mathbf{x} \in \mathbf{X}.$$

Let $\psi_i(\mathbf{x})$ be functionals on a linear space, E, such that $\psi_i(\mathbf{x})$ are convex for $i \leq 0$ and linear for $i \geq 0$ and $\mathbf{X}$ is some given convex subset of $\boldsymbol{E}$. Then in order that $\psi_0(\mathbf{x})$ achieves its minimum point at $\mathbf{x}^* \in E$ it is necessary that there exists constants $\lambda_i, \quad i = -m, ..., n$, such that

$$\sum_{i=-m}^{n} \lambda_i \psi_i(\mathbf{x}^*) \leq \sum_{i=-m}^{n} \lambda_i \psi_i(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{X}$. Moreover, $\lambda_i \geq 0$ for each $i \leq 0$, and $\lambda_i \psi_i(\mathbf{x}_0) = 0$ for each $i \neq 0$. If $\lambda_0 \geq 0$, then the conditions are also sufficient.

Let us write down the necessary and sufficient Kuhn-Tacker conditions for problems (P1),(P2), and (P3). After some equivalent transformations these conditions can be stated as follows:

Kuhn-Tacker conditions for (P1) are, actually, a definition of the minimum point.
K-T conditions for (P1)

$$\text{(KT1)} \qquad \phi_\beta(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \;\leq\; \phi_\beta(\mathbf{x}) - \mu_1 R(\mathbf{x})\,, \;\; \mu_1 \geq 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

K-T conditions for (P2)

$$\lambda_0^2 \phi_\beta(\mathbf{x}^*) + \lambda_1^2(\,\rho - R(\mathbf{x}^*)) \;\leq\; \lambda_0^2 \phi_\beta(\mathbf{x}) + \lambda_1^2(\,\rho - R(\mathbf{x}))\,,$$

$$\lambda_1^2(\,\rho - R(\mathbf{x})) = 0\,, \;\; \lambda_0^2 > 0\,, \;\; \lambda_1^2 \geq 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

$$\Downarrow$$

$$\text{(KT2)} \qquad \phi_\beta(\mathbf{x}^*) - \mu_2 R(\mathbf{x}^*) \ \le \ \phi_\beta(\mathbf{x}) - \mu_2 R(\mathbf{x})\,,$$

$$\mu_2(\,\rho - R(\mathbf{x}^*)) = 0\,,\ \ \mu_2 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

<u>K-T conditions for (P3)</u>

$$\lambda_0^3(-R(\mathbf{x}^*)) + \lambda_1^3(\phi_\beta(\mathbf{x}^*) - \omega) \ \le \ \lambda_0^3(-R(\mathbf{x})) + \lambda_1^3(\,\phi_\beta(\mathbf{x}) - \omega)\,,$$

$$\lambda_1^3(\phi_\beta(\mathbf{x}^*) - \omega) = 0\,,\ \ \lambda_0^3 > 0\,,\ \ \lambda_1^3 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

$$\Downarrow$$

$$\text{(KT3)} \qquad -R(\mathbf{x}^*) + \mu_3 \phi_\beta(\mathbf{x}^*) \ \le \ -R(\mathbf{x}) + \mu_3 \phi_\beta(\mathbf{x})\,,$$

$$\mu_3(\phi_\beta(\mathbf{x}^*) - \omega) = 0\,,\ \ \mu_3 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

Following [30], we call μ_2 in (KT2) the optimal reward multiplier, and μ_3 in (KT3) the risk multiplier. Further, using conditions (KT1) and (KT2), we show that a solution of problem (P1) is also a solution of (P2) and vice versa, a solution of problem (P2) is also a solution of (P1).

Lemma A1. *If a point $\mathbf{x}^*$ is a solution of (P1), then the point $\mathbf{x}^*$ is a solution of (P2) with parameter $\rho = R(\mathbf{x}^*)$. Also, stated in the other direction, if $\mathbf{x}^*$ is a solution of (P2) and μ_2 is the optimal reward multiplier in (KT2), then $\mathbf{x}^*$ is a solution of (P1) with $\mu_1 = \mu_2$.*

Proof of Lemma A1. Let us prove the first statement of Lemma A1. If $\mathbf{x}^*$ is a solution of (P1), then it satisfies condition (KT1). Evidently, this solution $\mathbf{x}^*$ satisfies (KT2) with $\rho = R(\mathbf{x}^*)$ and $\mu_2 = \mu_1$.

Now, let us prove the second statement of Lemma A1. Suppose that $\mathbf{x}^*$ is a solution of (P2) and (KT2) is satisfied. Then, (KT1) is satisfied with parameter $\mu_1 = \mu_2$ and $\mathbf{x}^*$ is a solution of (P1). Lemma A1 is proved. $\diamond$

Further, using conditions (KT1) and (KT3), we show that a solution of problems (P1) is also a solution of (P3) and vice versa, a solution of problems (P3) is also a solution of (P1).

Lemma A2. *If a point* $\mathbf{x}^*$ *is a solution of (P1), then the point* $\mathbf{x}^*$ *is a solution of (P3) with the parameter* $\omega = \phi_\beta(\mathbf{x})$*. Also, stated in other direction, if* $\mathbf{x}^*$ *is a solution of (P3) and* μ_3 *is a positive risk multiplier in (KT3), then* $\mathbf{x}^*$ *is a solution of (P1) with* $\mu_1 = 1/\mu_3$*.*

Proof of Lemma A2. Let us prove the first statement of Lemma A2. If $\mathbf{x}^*$ is a solution of (P1), then it satisfies the condition (KT1). If $\mu_1 > 0$, then this solution $\mathbf{x}^*$ satisfies (KT3) with $\mu_3 = 1/\mu_1$ and $\omega = \phi_\beta(\mathbf{x})$.

Now, let us prove the second statement of Lemma A2. Suppose that $\mathbf{x}^*$ is a solution of (P3) and (KT3) is satisfied with $\mu_3 > 0$. Then, (KT1) is satisfied with parameter $\mu_1 = 1/\mu_3$ and $\mathbf{x}^*$ is a solution of (P1). Lemma A2 is proved. $\diamond$

Lemma A1 implies that the efficient frontiers of problems (P1) and (P2) coincide. Similar, Lemma A2 implies that the efficient frontiers of problems (P1) and (P3) coincide. Consequently, efficient frontiers of problems (P1), (P2), and (P3) coincide. Theorem 3 is proved. $\diamond$

Appendix B. Proofs of Theorems 4 and 5.

Proof of Theorems 4. With Theorem A1, the necessary and sufficient conditions for the problem (P3') are stated as follows

$$\text{(KT3}'\text{)} \qquad -R(\mathbf{x}^*) + \mu_3 F_\beta(\mathbf{x}^*, \alpha^*) \ \leq\ -R(\mathbf{x}) + \mu_3 F_\beta(\mathbf{x}, \alpha)\,,$$

$$\mu_3(F_\beta(\mathbf{x}^*, \alpha^*) - \omega) = 0\,,\ \ \mu_3 \geq 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

First, suppose that $\mathbf{x}^*$ is a solution of (P3) and $\alpha^* \in A_\beta(\mathbf{x}^*)$. Let us show that $(\mathbf{x}^*, \alpha^*)$ is a solution of (P3'). Using necessary and sufficient conditions (KT3) and Theorem 1 we have

$$-R(\mathbf{x}^*) + \mu_3 F_\beta(\mathbf{x}^*, \alpha^*) \ =\ -R(\mathbf{x}^*) + \mu_3 \phi_\beta(\mathbf{x}^*)$$

$$\leq\ -R(\mathbf{x}) + \mu_3 \phi_\beta(\mathbf{x}) \ =\ -R(\mathbf{x}) + \mu_3 \min_\alpha F_\beta(\mathbf{x}, \alpha)$$

$$\leq\ -R(\mathbf{x}) + \mu_3 F_\beta(\mathbf{x}, \alpha)\,,$$

and

$$\mu_3(F_\beta(\mathbf{x}^*, \alpha^*) - \omega) = \mu_3(\phi_\beta(\mathbf{x}^*) - \omega) = 0\,,\ \ \mu_3 \geq 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

Thus, (KT3') conditions are satisfied and $(\mathbf{x}^*, \alpha^*)$ is a solution of (P3').

Now, let us suppose that $(\mathbf{x}^*, \alpha^*)$ achieves the minimum of (P3') and $\mu_3 > 0$. For fixed $\mathbf{x}^*$, the point α^* minimizes the function $-R(\mathbf{x}^*) + \mu_3 F_\beta(\mathbf{x}^*, \alpha)$, and, consequently, the function $F_\beta(\mathbf{x}^*, \alpha)$. Then, Theorem 1 implies that $\alpha^* \in A_\beta(\mathbf{x}^*)$. Further, since $(\mathbf{x}^*, \alpha^*)$ is a solution of (P3'), conditions (KT3') and Theorem 1 imply that

$$-R(\mathbf{x}^*) + \mu_3 \phi_\beta(\mathbf{x}^*) \ =\ -R(\mathbf{x}^*) + \mu_3 F_\beta(\mathbf{x}^*, \alpha^*)$$

$$\leq\ -R(\mathbf{x}) + \mu_3 F_\beta(\mathbf{x}, \alpha_\beta(\mathbf{x})) = -R(\mathbf{x}) + \mu_3 \phi_\beta(\mathbf{x})$$

and

$$\mu_3(\phi_\beta(\mathbf{x}^*) - \omega) = \mu_3(F_\beta(\mathbf{x}^*, \alpha^*) - \omega) = 0\,,\ \ \mu_3 \geq 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

We proved that conditions (KT3) are satisfied, i.e., $\mathbf{x}^*$ is a solution of (P3). Theorem 4 is proved. $\diamond$

Proof of Theorems 5. Let $\mathbf{x}^*$ is a solution of (P1), i.e.,

$$\phi_\beta(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \ \leq\ \phi_\beta(\mathbf{x}) - \mu_1 R(\mathbf{x})\,,\ \ \mu_1 \geq 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

and $\alpha^* \in A_\beta(\mathbf{x}^*)$. Using Theorem 1 we have

$$F_\beta(\mathbf{x}^*, \alpha^*) - \mu_1 R(\mathbf{x}^*) \; = \; \phi_\beta(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*)$$

$$\leq \; \phi_\beta(\mathbf{x}) - \mu_1 R(\mathbf{x}) \; = \; \min_{\alpha} F_\beta(\mathbf{x}, \alpha) - \mu_1 R(\mathbf{x})$$

$$\leq \; F_\beta(\mathbf{x}, \alpha) - \mu_1 R(\mathbf{x}) \, , \quad \mathbf{x} \in \mathbf{X} \, ,$$

i.e, $(\mathbf{x}^*, \alpha^*)$ is a solution of problem (P1').

Now, let us consider that $(\mathbf{x}^*, \alpha^*)$ is a solution of problem (P1'). For the fixed point $\mathbf{x}^*$, the point α^* minimizes the functions $F_\beta(\mathbf{x}^*, \alpha) - \mu_1 R(\mathbf{x}^*)$ and, consequently, the point α^* minimizes the function $F_\beta(\mathbf{x}^*, \alpha)$. Then, Theorem 1 implies that $\alpha^* \in A_\beta(\mathbf{x}^*)$. Further, since $(\mathbf{x}^*, \alpha^*)$ is a solution of (P1'), Theorem 1 implies

$$\phi_\beta(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \; = \; F_\beta(\mathbf{x}^*, \alpha^*) - \mu_1 R(\mathbf{x}^*)$$

$$\leq F_\beta(\mathbf{x}, \alpha_\beta(\mathbf{x})) - \mu_1 R(\mathbf{x}) \; = \; \phi_\beta(\mathbf{x}) - \mu_1 R(\mathbf{x}) \, , \quad \mathbf{x} \in \mathbf{X} \, .$$

Theorem 5 is proved. $\diamond$