Towards a Theory of Volatility Trading

Peter Carr
Morgan Stanley
1585 Broadway, 6th Floor
New York, NY 10036
(212) 761-7340
carrp@ms.com

Dilip Madan
College of Business and Management
University of Maryland
College Park, MD 20742
(301) 405-2127
dbm@mbs.umd.edu

Current Version: December 16, 1997

We thank the participants of presentations at Boston University, the NYU Courant Institute, M.I.T., Morgan Stanley, and the Risk 1997 Congress. We would also like to thank Marco Avellaneda, Joseph Cherian, Stephen Chung, Emanuel Derman, Raphael Douady, Bruno Dupire, Ognian Enchev, Chris Fernandes, Marvin Friedman Iraj Kani, Keith Lewis, Harry Mendell, Lisa Polsky, John Ryan, Murad Taqqu, Alan White, and especially Robert Jarrow for useful discussions. They are not responsible for any errors.

I Introduction

Much research has been directed towards forecasting the volatility[1] of various macroeconomic variables such as stock indices, interest rates and exchange rates. However, comparatively little research has been directed towards the optimal way to invest given a view on volatility. This absence is probably due to the belief that volatility is difficult to trade. For this reason, a small literature has emerged which advocates the development of volatility indices and the listing of financial products whose payoff is tied to these indices. For example, Gastineau[14] and Galai[13] propose the development of option indices similar in concept to stock indices. Brenner and Galai[4] propose the development of realized volatility indices and the development of futures and options contracts on these indices. Similarly, Fleming, Ostdiek and Whaley[12] describe the construction of an implied volatility index (the VIX), while Whaley[25] proposes derivative contracts written on this index. Grunbichler and Longstaff[16] develop a valuation model for options on volatility assuming a mean reverting volatility process.

In response to this hue and cry, some volatility contracts have been listed. For example, the OMLX, which is the London based subsidiary of the Swedish exchange OM, has launched volatility futures at the beginning of 1997. At this writing, the Deutsche Terminborse (DTB) recently launched its own futures based on its already established implied volatility index. Thus far, the volume in these contracts has been disappointing.

One possible explanation for this outcome is that volatility can already be traded by combining static positions in options on price with dynamic trading in the underlying. Neuberger[22] showed that by delta-hedging a contract paying the log of the price, the hedging error accumulates to the difference between the realized variance and the fixed variance used in the delta-hedge. The contract paying the log of the price can be created with a static position in options as shown in Breeden and Litzenberger[3]. Independently of Neuberger, Dupire[9] showed that a calendar spread of two such log contracts pays the variance between the 2 maturities, and developed the notion of forward variance. Following Heath, Jarrow

[1] In this article, the term "volatility" refers to either the variance or the standard deviation of the return on an investment.

and Morton[17](HJM), Dupire modelled the evolution of the term structure of this forward variance, thereby developing the first stochastic volatility model in which the market price of volatility risk does not require specification, even though volatility is imperfectly correlated with the price of the underlying.

The primary purpose of this article is to review three methods which have emerged for trading realized volatility. The first method reviewed involves taking static positions in options. The classic example is that of a long position in a straddle, since the value usually[2] increases with a rise in volatility. The second method reviewed involves delta-hedging an option position. If the investor is successful in hedging away the price risk, then a prime determinant of the profit or loss from this strategy is the difference between the realized volatility and the anticipated volatility used in pricing and hedging the option. The final method reviewed for trading realized volatility involves buying or selling an over-the-counter contract whose payoff is an explicit function of volatility. The simplest example of such a volatility contract is a vol swap. This contract pays the buyer the difference betweeen the realized volatility[3] and the fixed swap rate determined at the outset of the contract[4].

A secondary purpose of this article is to uncover the link between volatility contracts and some recent path-breaking work by Dupire[10] and by Derman, Kani, and Kamal[8](henceforth DKK). By restricting the set of times and price levels for which returns are used in the volatility calculation, one can synthesize a contract which pays off the "local volatility", i.e. the volatility which will be experienced should the underlying be at a specified price level at a specified future date. These authors develop the notion of forward local volatility, which is the fixed rate the buyer of the local vol swap pays at maturity in the event the specified price level is reached. Given a complete term and strike structure of options, the entire forward local volatility surface can be backed out from the prices of options. This surface is the two dimensional analog of the forward rate curve central to the HJM analysis. Following HJM, these authors impose a stochastic process on the forward local volatility surface and derive the risk-neutral dynamics of this surface.

[2]Jagannathan[18] shows that in general options need not be increasing in volatility.

[3]For marketing reasons, these contracts are usually written on the standard deviation, despite the focus of the literature on spanning contracts on variance.

[4]This contract is actually a forward contract on realized volatility, but is nonetheless termed a swap.

The outline of this paper is as follows. The next section looks at trading realized volatility via static positions in options. The theory of static replication using options is reviewed in order to develop some new positions for profiting from a correct view on volatility. The subsequent section shows how dynamic trading in the underlying can alternatively be used to create or hedge a volatility exposure. The fourth section looks at over-the-counter volatility contracts as a further alternative for trading volatility. The section shows how such contracts can be synthesized by combining static replication using options with dynamic trading in the underlying asset. A fifth section draws a link between these volatility contracts and the work on forward local volatility pioneered by Dupire and DKK. The final section summarizes and suggests some avenues for future research.

II Trading Realized Volatility via Static Positions in Options

The classic position for gaining exposure to volatility is to buy an at-the-money[5]straddle. Since at-the-money options are frequently used to trade volatility, the implied volatility from these options are widely used as a forecast of subsequent realized volatility. The widespread use of this measure is surprising since the approach relies on a model which itself assumes that volatility is constant.

This section derives an alternative forecast, which is also calculated from market prices of options. In contrast to implied volatility, the forecast does not assume constant volatility, or even that the underlying price process is continuous. In contrast to the implied volatility forecast, our forecast uses the market prices of options of all strikes. In order to develop the alternative forecast, the next subsection reviews the theory of static replication using options developed in Ross[24] and Breeden and Litzenberger[3]. The following subsection applies this theory to determine a model-free forecast of subsequent realized volatility.

[5]Note that in the Black model, the sensitivity to volatility of a straddle is actually maximized at slightly below the forward price.

II-A Static Replication with Options

Consider a single period setting in which investments are made at time 0 with all payoffs being received at time T. In contrast to the standard intertemporal model, we assume that there are no trading opportunities other than at times 0 and T. We assume there exists a futures market in a risky asset (eg. a stock index) for delivery at some date $T' \geq T$. We also assume that markets exist for European-style futures options[6] of all strikes. While the assumption of a continuum of strikes is far from standard, it is essentially the analog of the standard assumption of continuous trading. Just as the latter assumption is frequently made as a reasonable approximation to an environment where investors can trade frequently, our assumption is a reasonable approximation when there are a large but finite number of option strikes (eg. for S&P500 futures options).

It is widely recognized that this market structure allows investors to create any smooth function $f(F_T)$ of the terminal futures price by taking a static position at time 0 in options[7]. Appendix 1 shows that any twice differentiable payoff can be re-written as:

$$\begin{aligned} f(F_T) &= f(\kappa) + f'(\kappa)[(F_T - \kappa)^+ - (\kappa - F_T)^+] \\ &\quad + \int_0^{\kappa} f''(K)(K - F_T)^+ dK + \int_{\kappa}^{\infty} f''(K)(F_T - K)^+ dK. \end{aligned} \tag{1}$$

The first term can be interpreted as the payoff from a static position in $f(\kappa)$ pure discount bonds, each paying one dollar at T. The second term can be interpreted as the payoff from $f'(\kappa)$ calls struck at κ less $f'(\kappa)$ puts, also struck at κ. The third term arises from a static position in $f''(K)dK$ puts at all strikes less than κ. Similarly, the fourth term arises from a static position in $f''(K)dK$ calls at all strikes greater than κ.

In the absence of arbitrage, a decomposition similar to (1) must prevail among the initial values. Let V_0^f and B_0 denote the initial values of the payoff and the pure discount bond respectively. Similarly, let

[6] Note that listed futures options are generally American-style. However, by setting $T' = T$, the underlying futures will converge to the spot at T and so the assumption is that there exists European-style spot options in this special case.

[7] This observation was first noted in Breeden and Litzenberger[3] and established formally in Green and Jarrow[15] and Nachman[21].

$P_0(K)$ and $C_0(K)$ denote the initial prices of the put and the call struck at K respectively. Then the no arbitrage condition requires that:

$$
\begin{aligned}
V_0^f &= f(\kappa)B_0 + f'(\kappa)[C_0(\kappa) - P_0(\kappa)] \\
&\quad + \int_0^{\kappa} f''(K)P_0(K)dK + \int_{\kappa}^{\infty} f''(K)C_0(K)dK. \qquad (2)
\end{aligned}
$$

Thus, the value of an arbitary payoff can be obtained from bond and option prices. Note that no assumption was made regarding the stochastic process governing the futures price.

II-B An Alternative Forecast of Variance

Consider the problem of forecasting the variance of the log futures price relative $\ln\left(\frac{F_T}{F_0}\right)$. For simplicity, we refer to the log futures price relative as a return, even though no investment is required in a futures contract. The variance of the return over some interval $[0,T]$ is of course given by the expectation of the squared deviation of the return from its mean:

$$
\mathrm{Var}_0\left\{\ln\left(\frac{F_T}{F_0}\right)\right\} = E_0\left\{\ln\left(\frac{F_T}{F_0}\right) - E_0\left[\ln\left(\frac{F_T}{F_0}\right)\right]\right\}^2. \qquad (3)
$$

It is well-known that futures prices are martingales under the appropriate risk-neutral measure. When the futures contract marks to market continuously, then futures prices are martingales under the measure induced by taking the money market account as numeraire. When the futures contract marks to market daily, then futures prices are martingales under the measure induced by taking a daily rollover strategy as numeraire, where this strategy involves rolling over pure discount bonds with maturities of one day. Thus, given a mark-to-market frequency, futures prices are martingales under the measure induced by the rollover strategy with the same rollover frequency.

If the variance in (3) is calculated using this measure, then $E_0\left[\ln\left(\frac{F_T}{F_0}\right)\right]$ can be interpreted as the futures[8] price of a portfolio of options which pays off $f_m(F) \equiv \ln\left(\frac{F_T}{F_0}\right)$ at T. The spot value of this payoff

[8] Options do trade futures-style in Hong Kong. However, when only spot option prices are available, one can set $T' = T$ and calculate the mean and variance of the terminal spot under the forward measure. The variance is then expressed in terms of the forward prices of options, which can be obtained from the spot price by dividing by the bond price.

is given by (2) with κ arbitrary and $f''_m(K) = \frac{-1}{K^2}$. Setting $\kappa = F_0$, the futures price of the payoff is given by:

$$\mathcal{F} \equiv E_0\left[\ln\left(\frac{F_T}{F_0}\right)\right] = -\int_0^{F_0} \frac{1}{K^2}\hat{P}_0(K,T)dK - \int_{F_0}^{\infty} \frac{1}{K^2}\hat{C}_0(K,T)dK,$$

where $\hat{P}_0(K,T)$ and $\hat{C}_0(K,T)$ denote the initial futures price of the put and the call respectively, both for delivery at T. This futures price is initially negative[9] due to the concavity (negative time value) of the payoff.

Similarly, the variance of returns is just the futures price of the portfolio of options which pays off $f_v(F) = \left\{\ln\left(\frac{F_T}{F_0}\right) - \mathcal{F}\right\}^2$ at T (see Figure 1): The second derivative of this payoff is $f''_v(K) =$

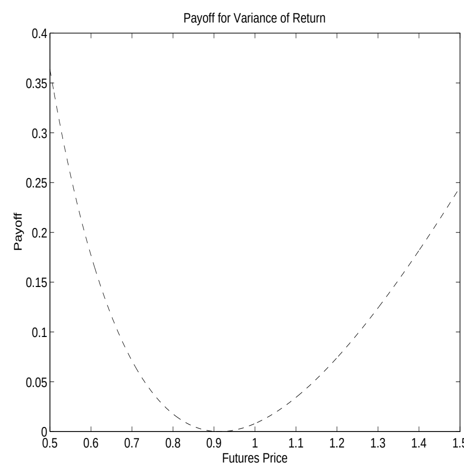

Figure 1: Payoff for Variance of Return ($F_0 = 1; \mathcal{F} = -.09$).

$\frac{2}{K^2}\left[1 - \ln\left(\frac{K}{F_0}\right) + \mathcal{F}\right]$. This payoff has zero value and slope at $F_0 e^{\mathcal{F}}$. Thus, setting $\kappa = F_0 e^{\mathcal{F}}$, the futures price of the payoff is given by:

$$\begin{aligned}\mathrm{Var}_0\left\{\ln\left(\frac{F_T}{F_0}\right)\right\} &= \int_0^{F_0 e^{\mathcal{F}}} \frac{2}{K^2}\left[1 - \ln\left(\frac{K}{F_0}\right) + \mathcal{F}\right]\hat{P}_0(K,T)dK \\ &\quad + \int_{F_0 e^{\mathcal{F}}}^{\infty} \frac{2}{K^2}\left[1 - \ln\left(\frac{K}{F_0}\right) + \mathcal{F}\right]\hat{C}_0(K,T)dK. \qquad (4)\end{aligned}$$

At time 0, this futures price is an interesting alternative to implied or historical volatility as a forecast of subsequent realized volatility. However, in common with any futures price, this forecast is a reflection

[9] If the futures price process is a continuous semi-martingale, then Itô's lemma implies that $E_0\left[\ln\left(\frac{F_T}{F_0}\right)\right] = -E_0 \frac{1}{2}\int_0^T \sigma_t^2 dt$, where σ_t is the volatility at time t.

of both statistical expected value and risk aversion. Consequently, by comparing this forecast with the ex-post outcome, the market price of variance risk can be inferred. We will derive a simpler forecast of variance in section IV under more restrictive assumptions, principally price continuity.

When compared to an at-the-money straddle, the static position in options used to create f_v has the advantage of maintaining sensitivity to volatility as the underlying moves away from its initial level. Unfortunately, like straddles, these contracts can take on significant price exposure once the underlying moves away from its initial level. An obvious solution to this problem is to delta-hedge with the underlying. The next section considers this alternative.

III Trading Realized Volatility by Delta-Hedging Options

The static replication results of the last section made no assumption whatsoever about the price process or volatility process. In order to apply delta-hedging with the underlying futures, we now assume that investors can trade continuously, that interest rates are constant, and that the underlying futures price process is a continuous semi-martingale. Note that we maintain our previous assumption that the volatility of the futures follows an arbitrary unknown stochastic process. While one could specify a stochastic process and develop the correct delta-hedge in such a model, such an approach is subject to significant model risk since one is unlikely to guess the correct volatility process. Furthermore, such models generally require dynamic trading in options which is costly in practice. Consequently, in what follows we leave the volatility process unspecified and restrict dynamic strategies to the underlying alone. Specifically, we assume that an investor follows the classic replication strategy specified by the Black model, with the delta calculated using a constant volatility σ_h. Since the volatility is actually stochastic[10],, the replication will be imperfect and the error results in either a profit or a loss realized at the expiration of the hedge.

To uncover the magnitude of this $P\&L$, let $V(F,t;\sigma)$ denote the Black model value of a European-style claim given that the current futures price is F and the current time is t. Note that the last argument of

[10] In an interesting paper, Cherian and Jarrow[7] show the existence of an equilibrium in an incomplete economy where investors believe the Black Scholes formula is valid even though volatility is stochastic.

V is the volatility used in the calculation of the value. In what follows, it will be convenient to have the attempted replication occur over an arbitrary future period (T, T') rather than over $(0, T)$. Consequently, we assume that the underlying futures matures at some date $T'' \geq T'$.

We suppose that an investor sells a European-style claim at T for the Black model value $V(F_T, T; \sigma_h)$ and holds $\frac{\partial V}{\partial F}(F_t, t; \sigma_h)$ futures contracts over (T, T'). Applying Itô's lemma to $V(F, t; \sigma_h)e^{r(T'-t)}$ gives:

$$\begin{aligned} V(F_{T'}, T'; \sigma_h) &= V(F_T, T; \sigma_h)e^{r(T'-T)} + \int_T^{T'} e^{r(T'-t)} \frac{\partial V}{\partial F}(F_t, t; \sigma_h) dF_t \\ &+ \int_T^{T'} e^{r(T'-t)} \left[-rV(F_t, t; \sigma_h) + \frac{\partial V}{\partial t}(F_t, t; \sigma_h) \right] dt + \int_T^{T'} e^{r(T'-t)} \frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h) \frac{F_t^2}{2} \sigma_t^2 dt. \end{aligned} \tag{5}$$

Now, by definition, $V(F, t; \sigma_h)$ solves the Black partial differential equation subject to a terminal condition:

$$-rV(F, t; \sigma_h) + \frac{\partial V}{\partial t}(F, t; \sigma_h) = -\frac{\sigma_h^2 F^2}{2} \frac{\partial^2 V}{\partial F^2}(F, t; \sigma_h), \tag{6}$$

$$V(F, T'; \sigma_h) = f(F). \tag{7}$$

Substituting (6) and (7) in (5) and re-arranging gives:

$$f(F_{T'}) + \int_T^{T'} e^{r(T'-t)} \frac{F_t^2}{2} \frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h)(\sigma_h^2 - \sigma_t^2) dt = V(F_T, T; \sigma_h)e^{r(T'-T)} + \int_T^{T'} e^{r(T'-t)} \frac{\partial V}{\partial F}(F_t, t; \sigma_h) dF_t. \tag{8}$$

The right hand side is clearly the terminal value of a dynamic strategy comprising an investment at T of $V(F_T, T; \sigma_h)$ dollars in the riskless asset and a dynamic position in $\frac{\partial V}{\partial F}(F_t, t; \sigma_h)$ futures contracts over the time interval (T, T'). Thus, the left hand side must also be the terminal value of this strategy, indicating that the strategy misses its target $f(F_{T'})$ by:

$$P\&L \equiv \int_T^{T'} e^{r(T'-t)} \frac{F_t^2}{2} \frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h)(\sigma_h^2 - \sigma_t^2) dt. \tag{9}$$

Thus, when a claim is sold for the implied volatility σ_h at T, the instantaneous P&L from delta-hedging it over (T, T') is $\frac{F_t^2}{2} \frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h)(\sigma_h^2 - \sigma_t^2)$, which is the difference between the hedge variance rate and the realized variance rate, weighted by half the dollar gamma. Note that the P&L (hedging error) will be zero if the realized instantaneous volatility σ_t is constant at σ_h. It is well known that claims with convex payoffs have nonnegative gammas ($\frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h) \geq 0$) in the Black model. For such claims (eg. options), if the hedge volatility is always less that the true volatility ($\sigma_h < \sigma_t$ for all $t \in [T, T']$), then a loss results, regardless of the path. Conversely, if the claim with a convex payoff is sold for an implied volatility σ_h which dominates[11] the subsequent realized volatility at all times, then delta-hedging at σ_h using the Black model delta guarantees a positive P&L.

When compared with static options positions, delta hedging appears to have the advantage of being insensitive to the price of the underlying. However, (9) indicates that the P&L at T' does depend on the final price as well as on the price path. An investor with a view on volatility alone would like to immunize the exposure to this path. One solution is to use a stochastic volatility model to conduct the replication of the desired volatility dependent payoff. However, as mentioned previously, this requires specifying a volatility process and employing dynamic replication with options. A better solution is to choose the payoff function $f(\cdot)$, so that the path dependence can be removed or managed. For example, Neuberger[22] recognized that if $f(F) = 2\ln F$, then $\frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_h) = e^{-r(T'-t)}\frac{-2}{F_t^2}$ and thus from (9), the P&L at T' is the payoff of a variance swap $\int_T^{T'}(\sigma_t^2 - \sigma_h^2)dt$. This volatility contract and others related to it are explored in the next section.

IV Trading Realized Volatility by Using Volatility Contracts

This section shows that several interesting volatility contracts can be manufactured by taking options positions and then delta-hedging them at zero volatility. Accordingly, suppose we set $\sigma_h = 0$ in (8) and negate both sides:

$$\int_T^{T'} \frac{F_t^2}{2} f''(F_t)\sigma_t^2 dt = f(F_{T'}) - f(F_T) - \int_T^{T'} f'(F_t)dF_t. \tag{10}$$

The left hand side is a payoff at T' based on both the realized instantaneous volatility σ_t^2 and the price path. The dependence of this payoff on f arises only through f'', and accordingly, we will henceforth only consider payoff functions f which have zero value and slope at a given point κ. The right hand side of (10) depends only on the price path and results from adding the following three payoffs:

[11]See El Karoui, Jeanblanc-Picque, and Shreve[11] for the extension of this result to the case when the hedger uses a delta-hedging strategy assuming that volatility is a function of stock price and time. Also see Avellaneda et. al.[1][2] and Lyons[20] for similar results.

1. The payoff from a static position in options maturing at T' paying $f(F_{T'})$ at T'.

2. The payoff from a static position in options maturing at T paying $-e^{-r(T'-T)}f(F_T)$ and future-valued to T'

3. The payoff from maintaining a dynamic position in $-e^{-r(T'-t)}f'(F_t)$ futures contracts over the time interval (T, T') (assuming continuous marking-to-market and that the margin account balance earns interest at the riskfree rate).

Thus, the payoff on the left-hand side can be achieved by combining a static position in options as discussed in section II, with a dynamic strategy in futures as discussed in section III. The dynamic strategy can be interpreted as an attempt to create the payoff $-f(F_{T'})$ at T', conducted under the false assumption of zero volatility. Since realized volatility will be positive, an error arises, and the magnitude of this error is given by $\int_T^{T'} \frac{F_t^2}{2} f''(F_t)\sigma_t^2 dt$, which is the left side of (10). The payoff $f(\cdot)$ can be chosen so that when its second derivative is substituted into this expression, the dependence on the path is consistent with the investor's joint view on volatility and price. In this section, we consider the following 3 second derivatives of payoffs at T' and work out the $f(\cdot)$ which leads to them:

Description of Payoff	$f''(F_t)$	Payoff at T'
Variance over Future Period	$\frac{2}{F_t^2}$	$\int_T^{T'} \sigma_t^2 dt$
Future Corridor Variance	$\frac{2}{F_t^2} 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]$	$\int_T^{T'} 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]\sigma_t^2 dt$
Future Variance Along Strike	$\frac{2}{\kappa^2}\delta(F_t - \kappa)$	$\int_T^{T'} \delta(F_t - \kappa)\sigma_t^2 dt.$

IV-A Contract Paying Future Variance

Consider the following payoff function $\phi(F)$ (see Figure 2):

$$\phi(F) \equiv 2\left[\ln\left(\frac{\kappa}{F}\right) + \frac{F}{\kappa} - 1\right], \tag{11}$$

where κ is an arbitrary finite positive number. The first derivative is given by:

$$\phi'(F) = 2\left[\frac{1}{\kappa} - \frac{1}{F}\right]. \tag{12}$$

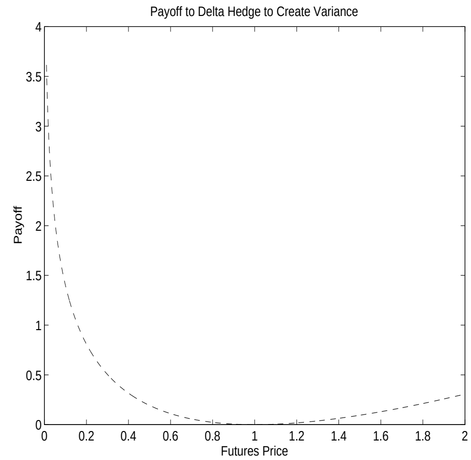

Figure 2: Payoff to Delta-Hedge to Create Contract Paying Variance ($\kappa = 1$).

Thus, the value and slope both vanish at $F = \kappa$. The second derivative of ϕ is simply:

$$\phi''(F) = \frac{2}{F^2}. \tag{13}$$

Substituting (11) to (13) into (10) results in a relationship between a contract paying the realized variance over the time interval (T, T') and three payoffs based on price:

$$\int_T^{T'} \sigma_t^2 dt = 2\left[\ln\left(\frac{\kappa}{F_{T'}}\right) + \frac{F_{T'}}{\kappa} - 1\right] - 2\left[\ln\left(\frac{\kappa}{F_T}\right) + \frac{F_T}{\kappa} - 1\right] - 2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{F_t}\right] dF_t. \tag{14}$$

The first two terms on the right hand side arise from static positions in options. Substituting (13) into (2) implies that for each term, the required position is given by:

$$2\left[\ln\left(\frac{\kappa}{F}\right) + \frac{F}{\kappa} - 1\right] = \int_0^{\kappa} \frac{2}{K^2}(K - F)^+ dK + \int_{\kappa}^{\infty} \frac{2}{K^2}(F - K)^+ dK, \tag{15}$$

Thus, to create the contract paying $\int_T^{T'} \sigma_t^2 dt$ at T', at $t = 0$, the investor should buy options at the longer maturity T' and sell options at the nearer maturity T. The inital cost of this position is given by:

$$\begin{aligned} &\textstyle\int_0^{\kappa} \frac{2}{K^2} P_0(K, T')dK + \int_{\kappa}^{\infty} \frac{2}{K^2} C_0(K, T')dK \\ &-e^{-r(T'-T)}[\textstyle\int_0^{\kappa} \frac{2}{K^2} P_0(K, T)dK + \int_{\kappa}^{\infty} \frac{2}{K^2} C_0(K, T)dK]. \end{aligned} \tag{16}$$

When the nearer maturity options expire, the investor should borrow to finance the payout of $2e^{-r(T'-T)}\left[\ln\left(\frac{\kappa}{F_T}\right) + \frac{F_T}{\kappa} - 1\right]$. At this time, the investor should also start a dynamic strategy in futures, holding $-2e^{-r(T'-t)}\left[\frac{1}{\kappa}-\frac{1}{F_t}\right]$ futures contracts for each $t \in [T, T']$. The net payoff at T' is:

$$2\left[\ln\left(\frac{\kappa}{F_{T'}}\right)+\frac{F_{T'}}{\kappa}-1\right]-2\left[\ln\left(\frac{\kappa}{F_T}\right)+\frac{F_T}{\kappa}-1\right]-2\int_T^{T'}\left[\frac{1}{\kappa}-\frac{1}{F_t}\right]dF_t=\int_T^{T'}\sigma_t^2 dt,$$

as required. Since the initial cost of achieving this payoff is given by (16), an interesting forecast $\hat{\sigma}^2_{T,T'}$ of the variance between T and T' is given by the future value of this cost:

$$\begin{aligned}\hat{\sigma}^2_{T,T'} &= e^{rT'}\int_0^{\kappa}\frac{2}{K^2}P_0(K,T')dK+\int_{\kappa}^{\infty}\frac{2}{K^2}C_0(K,T')dK \\ &\quad -e^{rT}\left[\int_0^{\kappa}\frac{2}{K^2}P_0(K,T)dK+\int_{\kappa}^{\infty}\frac{2}{K^2}C_0(K,T)dK\right].\end{aligned}$$

In contrast to implied volatility, this forecast does not use a model in which volatility is assumed to be constant. However, in common with any forward price, this forecast is a reflection of both statistical expected value and risk aversion. Consequently, by comparing this forecast with the ex-post outcome, the market price of volatility risk can be inferred.

IV-B Contract Paying Future Corridor Variance

In this subsection, we generalize to a contract which pays the "corridor variance", defined as the variance calculated using only the returns at times for which the futures price is within a specified corridor. In particular, consider a corridor $(\kappa-\triangle\kappa, \kappa+\triangle\kappa)$ centered at some arbitrary level κ and with width $2\triangle\kappa$. Suppose that we wish to generate a payoff at T' of $\int_T^{T'} 1[F_t \in (\kappa-\triangle\kappa, \kappa+\triangle\kappa)]\sigma_t^2 dt$. Thus, the variance calculation is based only on returns at times in which the futures price is inside the cooridot.

Consider the following payoff $\phi_{\triangle\kappa}(\cdot)$:

$$\phi_{\triangle\kappa}(F) \equiv 2\left[\ln\left(\frac{\kappa}{\underline{\bar{F}}}\right)+F\left(\frac{1}{\kappa}-\frac{1}{\underline{\bar{F}}}\right)\right], \tag{17}$$

where:

$$\underline{\bar{F}}_t \equiv \max[\kappa-\triangle\kappa, \min(F_t, \kappa+\triangle\kappa)]$$

is the futures price floored at $\kappa-\triangle\kappa$ and capped at $\kappa+\triangle\kappa$ (see Figure 3):

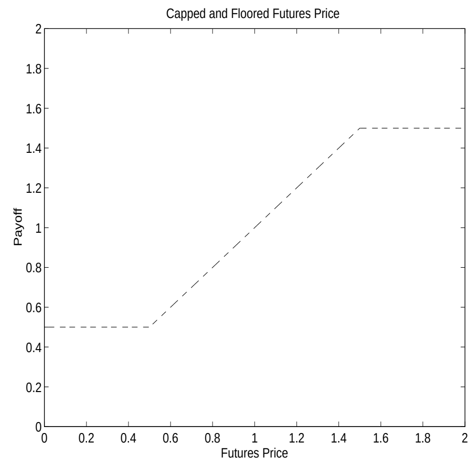

Figure 3: Futures Price Capped and Floored($\kappa = 1, \triangle\kappa = 0.5$).

From inspection, the payoff $\phi_{\triangle\kappa}(\cdot)$ is the same as ϕ defined in (11), but with F replaced by $\bar{\underline{F}}$. The new payoff is graphed in Figure 4: This payoff is actually a generalization of (11) since $\lim\limits_{\triangle\kappa\uparrow\infty} \bar{\underline{F}} = F$. For a finite

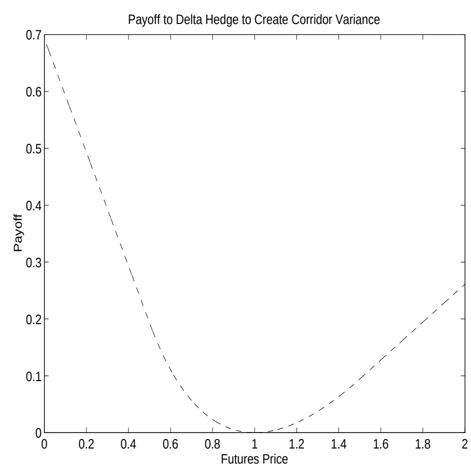

Figure 4: Trimming the Log Payoff ($\kappa = 1, \triangle\kappa = 0.5$).

corridor width, the payoff $\phi_{\triangle\kappa}(F)$ matches $\phi(F)$ for futures prices within the corridor. Consequently, like $\phi(F)$, $\phi_{\triangle\kappa}(F)$ has zero value and slope at $F = \kappa$. However, in contrast to $\phi(F)$, $\phi_{\triangle\kappa}(F)$ is linear outside the corridor with the lines chosen so that the payoff is continuous and differentiable at $\kappa \pm \triangle\kappa$. The first derivative of (17) is given by:

$$\phi'_{\triangle\kappa}(F) = 2\left[\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}}\right], \tag{18}$$

while the second derivative is simply:

$$\phi''_{\triangle\kappa}(F) = \frac{2}{F^2} 1[F \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]. \tag{19}$$

Substituting (17) to (19) into (10) implies that the volatility-based payoff decomposes as:

$$\begin{aligned}\int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt &= 2\left[\ln\left(\frac{\kappa}{\bar{\underline{F}}_{T'}}\right) + F_{T'}\left(\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_{T'}}\right)\right] - 2\left[\ln\left(\frac{\kappa}{\bar{\underline{F}}_T}\right) + F_T\left(\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_T}\right)\right] \\ &\quad -2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_t}\right] dF_t.\end{aligned}$$

The payoff function $\phi_{\triangle\kappa}(\cdot)$ has no curvature outside the corridor and consequently, the static positions in options needed to create the first two terms will not require strikes set outside the corridor. Thus, to create the contract paying the future corridor variance, $\int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt$ at T', the investor should initially only buy and sell options struck within the corridor, for an initial cost of:

$$\begin{aligned}&\textstyle\int_{\kappa-\triangle\kappa}^{\kappa} \frac{2}{K^2} P_0(K,T')dK + \int_{\kappa}^{\kappa+\triangle\kappa} \frac{2}{K^2} C_0(K,T')dK \\ &-e^{-r(T'-T)}[\textstyle\int_{\kappa-\triangle\kappa}^{\kappa} \frac{2}{K^2} P_0(K,T)dK + \int_{\kappa}^{\kappa+\triangle\kappa} \frac{2}{K^2} C_0(K,T)dK].\end{aligned}$$

At $t = T$, the investor should borrow to finance the payout of $2e^{-r(T'-T)}\left[\ln\left(\frac{\kappa}{\bar{\underline{F}}_T}\right) + F_T\left(\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_T}\right)\right]$ from having initially written the T maturity options. The investor should also start a dynamic strategy in futures, holding $-2e^{-r(T'-t)}\left[\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_t}\right]$ futures contracts for each $t \in [T, T']$. This strategy is semi-static in that no trading is required when the futures price is outside the corridor. The net payoff at T' is:

$$\begin{aligned}&2\left[\ln\left(\frac{\kappa}{\bar{\underline{F}}_{T'}}\right) + F_{T'}\left(\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_{T'}}\right)\right] - 2\left[\ln\left(\frac{\kappa}{\bar{\underline{F}}_T}\right) + F_T\left(\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_T}\right)\right] \\ &-2\int_T^{T'} \left[\frac{1}{\kappa} - \frac{1}{\bar{\underline{F}}_t}\right] dF_t = \int_T^{T'} \sigma_t^2 1[F_t \in (\kappa - \triangle\kappa, \kappa + \triangle\kappa)]dt,\end{aligned}$$

as desired.

IV-C Contract Paying Future Variance Along a Strike

In the last subsection, only options struck within the corridor were used in the static options position, and dynamic trading in the underlying futures was required ony when the futures price was in the corridor.

In this subsection, we shrink the width of the corridor of the last subsection down to a single point and examine the impact on the volatility based payoff and its replicating strategy. In order that this payoff have a non-negligible value, all asset positions in subsection IV-B must be re-scaled by $\frac{1}{2\triangle\kappa}$. Thus, the volatility-based payoff at T' would instead be $\int_T^{T'} \frac{1[F_t \in (\kappa-\triangle\kappa, \kappa+\triangle\kappa)]}{2\triangle\kappa}\sigma_t^2 dt$. By letting $\triangle\kappa \downarrow 0$, the variance received can be completely localized in the spatial dimension to $\int_T^{T'} \delta(F_t - \kappa)\sigma_t^2 dt$, where $\delta(\cdot)$ denotes a Dirac delta function[12]. Recalling that only options struck within the corridor are used to create the corridor variance, the initial cost of creating this localized cash flow is given by the following ratioed calendar spread of straddles:

$$\frac{1}{\kappa^2}[V_0(\kappa, T') - e^{-r(T'-T)}V_0(\kappa, T)],$$

where $V_0(\kappa, T)$ is the initial cost of a straddle struck at κ and maturing at T:

$$V_0(\kappa, T) \equiv P_0(\kappa, T) + C_0(\kappa, T).$$

As usual, at $t = T$, the investor should borrow to finance the payout of $\frac{|F_T - \kappa|}{\kappa^2}$ from having initially written the T maturity straddle. The appendix proves that the dynamic strategy in futures initiated at T involves holding $-\frac{e^{-r(T'-t)}}{\kappa^2} sgn(F_t - \kappa)$ futures contracts, where $sgn(x)$ is the sign function:

$$sgn(x) \equiv \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0. \end{cases}$$

When $T = 0$, this strategy reduces to the initial purchase of a straddle maturing at T', initially borrowing $e^{-rT'}|F_0 - \kappa|$ dollars and holding $-\frac{e^{-r(T'-t)}}{\kappa^2} sgn(F_t - \kappa)$ futures contracts for $t \in (0, T')$. The component of this strategy involving borrowing and futures is known as the stop-loss start-gain strategy, previously investigated by Carr and Jarrow[5]. By the Tanaka-Meyer formula[13], the difference between the payoff from the straddles and this dynamic strategy is known as the *local time* of the futures price process. Local time is a fundamental concept in the study of one dimensional stochastic processes. Fortunately, a straddle

[12]The Dirac delta function is a generalized function characterized by two properties:

1. $\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$
2. $\int_{-\infty}^{\infty} \delta(x) dx = 1$.

. See Richards and Youn[23] for an accessible introduction to such generalized functions.

[13]See Karatzas and Shreve[19], pg. 220.

combined with a stop-loss start-gain strategy in the underlying provides a mechanism for synthesizing a contract paying off this fundamental concept. The initial time value of the straddle is the market's (risk-neutral) expectation of the local time. By comparing this time value with the ex-post outcome, the market price of local time risk can be inferred.

V Connection to Recent Work on Stochastic Volatility

The last contract examined in the last section represents the limit of a localization in the futures price. When a continuum of option maturities is also available, we may additionally localize in the time dimension as has been done in some recent work by Dupire[10] and DKK[8]. Accordingly, suppose we further re-scale all the asset positions described in subsection IV-C by $\frac{1}{\triangle T}$ where $\triangle T \equiv T' - T$. The payoff at T' would instead be:

$$\int_T^{T'} \frac{\delta(F_t - \kappa)}{\triangle T} \sigma_t^2 dt.$$

The cost of creating this position would be:

$$\frac{1}{\kappa^2}\left[\frac{V_0(\kappa, T') - e^{-r(T'-T)} V_0(\kappa, T)}{\triangle T}\right].$$

By letting $\triangle T \downarrow 0$, one gets the beautiful result of Dupire[10] that $\frac{1}{\kappa^2}\left[\frac{\partial V_0}{\partial T}(\kappa, T) + r V_0(\kappa, T)\right]$ is the cost of creating the payment $\delta(F_T - \kappa)\sigma_T^2$ at T. As shown in Dupire, the forward local variance can be defined as the number of butterfly spreads paying $\delta(F_T - \kappa)$ at T one must sell in order to finance the above option position initially. A discretized version of this result can be found in DKK[8]. One can go on to impose a stochastic process on the forward local variance as in Dupire[10] and in DKK[8]. These authors derive conditions on the risk-neutral drift of the forward local variance, allowing replication of price or volatility-based payoffs using dynamic trading in only the underlying asset and a single option[14]. In contrast to earlier work on stochastic volatility, the form of the market price of volatility risk need not be specified.

[14]When two Brownian motions drive the price and the forward local volatility surface, any two assets whose payoffs are not co-linear can be used to span.

Summary and Suggestions for Future Research

We reviewed three approaches for trading volatility. While static positions in options do generate exposure to volatility, they also generate exposure to price. Similarly, a dynamic strategy in futures alone can yield a volatility exposure, but always has a price exposure as well. By combining static positions in options with dynamic trading in futures, payoffs related to realized volatility can be achieved which have either no exposure to price, or which have an exposure contingent on certain price levels being achieved in specified time intervals.

Under certain assumptions, we were able to price and hedge certain volatility contracts without specifying the process for volatility. The principle assumption made was that of price continuity. Under this assumption, a calendar spread of options emerges as a simple tool for trading the local volatility (or local time) between the two maturities. It would be interesting to see if this insight survives the relaxation of the critical assumption of price continuity. It would also be interesting to consider contracts which pay nonlinear functions of realized variance or local variance. Finally, it would be interesting to develop contracts on other statistics of the sample path such as the Sharpe ratio, skewness, covariance, correlation, etc. In the interests of brevity, such inquiries are best left for future research.

References

[1] Avellaneda, M., A. Levy, and A. Paras, 1995, "Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities", *Applied Mathematical Finance*, **2**, 73–88.

[2] Avellaneda, M., A. Levy, and A. Paras, 1996, "Managing the Volatility Risk of Portfolios of Derivative Securities: The Lagrangian Uncertain Volatility Model", *Applied Mathematical Finance*, **3**, 21–52.

[3] Breeden, D. and R. Litzenberger, 1978, "Prices of State Contingent Claims Implicit in Option Prices," *Journal of Business*, **51**, 621-651.

[4] Brenner, M., and D. Galai, 1989, "New Financial Instruments for Hedging Changes in Volatility", *Financial Analyst's Journal*, July-August 1989, 61–65.

[5] Carr P. and R. Jarrow, 1990, "The Stop-Loss Start-Gain Strategy and Option Valuation: A New Decomposition into Intrinsic and Time Value", *Review of Financial Studies*, **3**, 469–492.

[6] Carr P. and D. Madan, 1997, "Optimal Positioning in Derivative Securities", Morgan Stanley working paper.

[7] Cherian, J., and R. Jarrow, 1997, "Options Markets, Self-fulfilling Prophecies and Implied Volatilities", Boston University working paper, forthcoming in *Review of Derivatives Research.*

[8] Derman E., I. Kani, and M. Kamal, 1997, "Trading and Hedging Local Volatility" *Journal of Financial Engineering*, **6**, 3, 233-68.

[9] Dupire B., 1993, "Model Art", *Risk*. Sept. 1993, pgs. 118 and 120.

[10] Dupire B., 1996, "A Unified Theory of Volatility", Paribas working paper.

[11] El Karoui, N., M. Jeanblanc-Picque, and S. Shreve, 1996, "Robustness of the Black and Scholes Formula", Carnegie Mellon University working paper.

[12] Fleming, J., B. Ostdiek, and R. Whaley, 1993, "Predicting Stock Market Volatility: A New Measure", Duke University working paper.

[13] Galai, D., 1979, "A Proposal for Indexes for Traded Call Options", *Journal of Finance*, XXXIV, 5, 1157–72.

[14] Gastineau, G., 1977, "An Index of Listed Option Premiums", *Financial Analyst's Journal*, May-June 1977.

[15] Green, R.C. and R.A. Jarrow, 1987, "Spanning and Completeness in markets with Contingent Claims," *Journal of Economic Theory*, **41**, 202-210.

[16] Grunbichler A., and F. Longstaff, 1993, "Valuing Options on Volatility", UCLA working paper.

[17] Heath, D., R. Jarrow, and A. Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation", *Econometrica*, **66** 77–105.

[18] Jagannathan R., 1984, "Call Options and the Risk of Underlying Securities", *Journal of Financial Economics*, **13**, 3, 425–434.

[19] Karatzas, I., and S. Shreve, 1988, <u>Brownian Motion and Stochastic Calculus</u>, Springer Verlag, NY.

[20] Lyons, T., 1995, "Uncertain Volatility and the Risk-free Synthesis of Derivatives", *Applied Mathematical Finance*, **2**, 117-33.

[21] Nachman, D., 1988, "Spanning and Completeness with Options," *Review of Financial Studies*, **3**, 31, 311-328.

[22] Neuberger, A. 1990, "Volatility Trading", London Business School working paper.

[23] Richards, J.I., and H.K. Youn, 1990 <u>Theory of Distributions: A Non-technical Introduction</u>, Cambridge University Press, 1990.

[24] Ross, S., 1976, "Options and Efficiency", *Quarterly Journal of Economics*, **90** Feb. 75–89.

[25] Whaley, R., 1993, "Derivatives on Market Volatility: Hedging Tools Long Overdue", *The Journal of Derivatives*, Fall 1993, 71–84.

Appendix 1: Spanning with Bonds and Options

For any payoff $f(F)$, the sifting property of a Dirac delta function implies:

$$\begin{aligned} f(F) &= \int_0^\infty f(K)\delta(F-K)dK \\ &= \int_0^\kappa f(K)\delta(F-K)dK + \int_0^\kappa f(K)\delta(F-K)dK, \end{aligned}$$

for any nonnegative κ. Integrating each integral by parts implies:

$$\begin{aligned} f(F) &= f(K)1(F<K)\Big|_0^\kappa - \int_0^\kappa f'(K)1(F<K)dK \\ &\quad + f(K)1(F \geq K)\Big|_\kappa^\infty + \int_\kappa^\infty f'(K)1(F \geq K)dK. \end{aligned}$$

Integrating each integral by parts once more implies:

$$\begin{aligned} f(F) &= f(\kappa)1(F<\kappa) - f'(K)(K-F)^+\Big|_0^\kappa + \int_0^\kappa f''(K)(K-F)^+dK \\ &\quad + f(\kappa)1(F \geq \kappa) - f'(K)(F-K)^+\Big|_\kappa^\infty + \int_\kappa^\infty f''(K)(F-K)^+dK \\ &= f(\kappa) + f'(\kappa)[(F-\kappa)^+ - (\kappa-F)^+] \\ &\quad + \int_0^\kappa f''(K)(K-F)^+dK + \int_\kappa^\infty f''(K)(F-K)^+dK. \end{aligned}$$

Appendix 2: Derivation of Futures Position When Synthesizing Contract Paying Future Variance Along a Strike

Recall from section IV-C, that all asset positions in section IV-B were normalized by multiplying by $\frac{1}{2\triangle\kappa}$. Thus in particular, the futures position of $-2e^{-r(T'-t)}\left[\frac{1}{\kappa}-\frac{1}{\bar{F}_t}\right]$ contracts in subsection IV-B is changed to $-\frac{e^{-r(T'-t)}}{\triangle\kappa}\left[\frac{1}{\kappa}-\frac{1}{\bar{F}_t}\right]$ contracts in subsection IV-C. More explicitly, the number of contracts held is given by

$$\begin{cases} -\frac{e^{-r(T'-t)}}{\triangle\kappa}\left[\frac{1}{\kappa}-\frac{1}{\kappa-\triangle\kappa}\right] & \text{if } F_t \leq \kappa-\triangle\kappa; \\ -\frac{e^{-r(T'-t)}}{\triangle\kappa}\left[\frac{1}{\kappa}-\frac{1}{F_t}\right] & \text{if } F_t \in (\kappa-\triangle\kappa,\kappa+\triangle\kappa); \\ -\frac{e^{-r(T'-t)}}{\triangle\kappa}\left[\frac{1}{\kappa}-\frac{1}{\kappa+\triangle\kappa}\right] & \text{if } F_t \geq \kappa+\triangle\kappa. \end{cases}$$

Now, by Taylor's series:

$$\frac{1}{\kappa-\triangle\kappa}=\frac{1}{\kappa}+\frac{1}{\kappa^2}\triangle\kappa+O(\triangle\kappa^2)$$

and:

$$\frac{1}{\kappa+\triangle\kappa}=\frac{1}{\kappa}-\frac{1}{\kappa^2}\triangle\kappa+O(\triangle\kappa^2).$$

Substitution implies that the number of futures contracts held is given by:

$$\begin{cases} -\frac{e^{-r(T'-t)}}{\triangle\kappa}\left[-\frac{1}{\kappa^2}\triangle\kappa+O(\triangle\kappa^2)\right] & \text{if } F_t \leq \kappa-\triangle\kappa; \\ -\frac{e^{-r(T'-t)}}{\triangle\kappa}\left[\frac{1}{\kappa}-\frac{1}{F_t}\right] & \text{if } F_t \in (\kappa-\triangle\kappa,\kappa+\triangle\kappa); \\ -\frac{e^{-r(T'-t)}}{\triangle\kappa}\left[\frac{1}{\kappa^2}\triangle\kappa+O(\triangle\kappa^2)\right] & \text{if } F_t \geq \kappa+\triangle\kappa. \end{cases}$$

Thus, as $\triangle\kappa \downarrow 0$, the number of futures contracts held converges to $-\frac{e^{-r(T'-t)}}{\kappa^2}sgn(F_t-\kappa)$, where $sgn(x)$ is the sign function:

$$sgn(x) \equiv \begin{cases} -1 & \text{if } x<0; \\ 0 & \text{if } x=0; \\ 1 & \text{if } x>0. \end{cases}$$