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**Derivatives on Volatility:**
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**Abstract:** Proposals to introduce derivatives whose payouts are explicitly linked to the volatility of an underlying asset have been around for some time. In response to these proposals, a few papers have tried to develop valuation formulae for volatility derivatives—derivatives that essentially help investors hedge the unpredictable volatility risk. This paper contributes to this nascent literature by developing closed-form/analytical formulae for prices of options and futures on volatility as well as volatility swaps. The primary contribution of this paper is that, unlike all other models, our model is empirically viable and can be easily implemented.

More specifically, our model distinguishes itself from other proposed solutions/models in the following respects: (1) Although volatility is stochastic, it is an exact function of the observed path of asset prices. This is crucial in practice because nonobservability of volatility makes it very difficult (in fact, impossible) to arrive at prices and hedge ratios of volatility derivatives in an internally consistent fashion, as it is akin to not knowing the stock price when trying to price an equity derivative. (2) The model does not require an unobserved volatility risk premium, nor is it predicated on the strong assumption of the existence of a continuum of options of all strikes and maturities as in some papers. (3) We show how it is possible to replicate (delta hedge) volatility derivatives by trading only in the underlying asset (on whose volatility the derivative exists) and a risk-free asset. This bypasses the problem of having to trade numerously many options on the underlying asset, a hedging strategy proposed in some other models.

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Key words: volatility, options, hedge

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**Derivatives on Volatility: Some Simple Solutions Based on Observables**

It is well known that volatility of most asset prices evolves unpredictably through time just as the level of asset price does. However, often volatility risk is difficult to hedge. As a result it has often been suggested that derivatives whose payoffs are explicitly tied to volatility be introduced. This motivation has been discussed in several papers. Brenner and Galai (1989) suggest that volatility indices (similar to the stock/equity indices) be constructed and options and futures on these indices be introduced. Whaley (1993) proposes derivatives contracts written on the implied volatility index of the Chicago Board Options Exchange (CBOE). As of now, many volatility derivatives, such as volatility swaps and options on volatility are traded over-the-counter. Many of these contracts have experienced sharp increases in trading volume in the aftermath of the financial crises of 1997 and 1998 that gave rise to very high volatility levels. Valuation formulae for volatility derivatives have been proposed in Grunbichler and Longstaff (1996), Whaley (1993). In a somewhat different approach to this subject, Carr and Madan (1998), Derman and Kani (1997) and Dupire (1997) have shown how the future level of volatility can be inferred from the prices of traded options of the underlying asset and thereby derivatives on volatility be valued.

However each of these models has features that make it difficult to implement them in practice. Specifying a process for volatility in continuous time, for example the mean reverting square root process of Grunbichler and Longstaff (1996) has the disadvantage that the underlying on which the derivative exists is unobservable. Further a functional form for the volatility risk premium has to be specified (unless it is assumed that investors do not demand any compensation for volatility). An unobserved volatility process along with an unobserved volatility risk-premium makes it very difficult to implement the model in practice. It is similar to not knowing the price of a stock when trying to value an equity option. On the other hand, trying to value an option on the implied volatility index of the CBOE (by specifying a stochastic process for the volatility index) as in Whaley (1993) faces the problem that the implied volatility index is inferred from the Black-Scholes model in which volatility is deterministic. In a somewhat different approach Carr and Madan (1998) (see Derman and Kani (1997) and Dupire

(1997) for related approaches) show that one does not necessarily require a volatility risk premium for valuing volatility derivatives. Also the valuation of certain volatility contracts such as futures and swaps on volatility can be independent of any functional specification for the volatility process. Although interesting, these approaches do require that a continuum of options of every strike and maturity on the underlying asset be traded to span volatility. Computing the value of volatility derivatives using this approach would require one to use numerous interpolations and extrapolations from a very limited number of options strikes (5 to 15 points apart in very actively traded options such as options on the S&P 500 index and more so in other markets) to infer the prices of an entire continuum of options on the underlying asset. Furthermore, it is not clear how one would estimate the parameters of the model in some of these frameworks (e.g. Derman and Kani (1997)) once a process for the local volatility process (associated with every strike and maturity) needs to be specified to value certain types of volatility derivatives such as options on volatility. More importantly, replicating/hedging volatility derivatives in these frameworks would require trading in options on the underlying asset that do not exist at all. Further the cost of replication would be prohibitively high given the numerous options that one has to trade and the high bid-ask spreads in most options markets.

This paper develops analytical solutions (both in discrete and continuous time) for volatility derivatives such as options and futures on volatility as well as volatility swaps. For options, we develop solutions for two types of scenarios: one where the payoff at maturity is tied to the variance at maturity and a second contract where the payoff is tied to the average (arithmetic average) level of the variance until maturity. Our model distinguishes itself from the above approaches in many dimensions. First of all, the level of volatility in our model is explicitly observable as a function of the path of the discretely observed history of asset prices. In other words, the underlying on which the option exists is known exactly at each point of time and all model parameters are easily estimated. As a result the model can be easily implemented solely by observing the history of asset price (price of an asset whose volatility we are interested in) observed at discrete intervals. Also our model does not require that the unobservable volatility risk premium be known. Unlike Carr and Madan (1998) we do not require the existence of a continuum of options of all strikes and maturities to compute prices of volatility derivatives.

Instead, the model is easily implemented just by observing the history of asset prices. For the continuous time model, we show how it is possible to replicate (delta hedge) volatility derivatives by trading only in the underlying asset (on whose volatility the option exists) or a futures contract on the underlying asset and a risk free asset without trading in an infinite number (or numerously many) options as required in Carr and Madan (1998), Derman and Kani (1997) and Dupire (1997). This is a significant advantage as volatility in itself cannot be traded and as mentioned before, trading even only a finite number of options is very costly given that the transaction costs/bid-ask spreads in many options markets are quite high. Also implementing the continuous time model is straightforward as the parameters of the continuous time model can be arbitrarily approximated by the parameters of the discrete time model that are easily estimated.

The analytical solutions developed in this paper are based on a discrete-time GARCH volatility process (and its continuous time counterpart) developed in Heston and Nandi (2000). While we do require a parametric specification for the volatility process, Heston and Nandi have shown that the volatility process used in this paper fits the volatility dynamics in many markets (e.g. S&P 500 index) very well. Furthermore an option pricing model for the underlying asset based on this volatility process has been empirically shown (see Heston and Nandi (2000)) to dominate many extant option pricing models, including a heuristic procedure often employed by traders to update their estimates of volatility by fitting an *ad hoc* version of the Black-Scholes model to the skew in implied volatilities and the term structure of volatility in the S&P 500 index options market (see Dumas, Fleming and Whaley (1998) for the *ad hoc* Black-Scholes model).

Section 1 describes the basic discrete-time model, Section 2 describes the continuous-time model and describes the construction of the replicating portfolio, while Section 3 concludes. The appendix contains detailed calculations and derivations of the option formula.

## 1. Discrete-Time Model

The discrete-time model has two basic assumptions. The first assumption is that the logarithm of the price of the underlying asset (i.e. the spot price) follows a particular GARCH process.

Assumption 1: The spot asset price, S(t) (including accumulated interest or dividends) follows the following process over time steps of length $\Delta$,

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}z(t), \qquad (1)$$
$$h(t) = \omega + \beta h(t-\Delta) + \alpha\left(z(t-\Delta)-\gamma\sqrt{h(t-\Delta)}\right)^2,$$

where r is the continuously compounded interest rate for the time interval $\Delta$ and z(t) is a standard normal disturbance. h(t) is the conditional variance of the log return between t - $\Delta$ and t and is known from the information set at time t - $\Delta$. The conditional variance in equation (1), although functionally different from some existing GARCH models, in fact is similar to the NGARCH and VGARCH models of Engle and Ng (1993). The conditional variance h(t) appears in the mean as a return premium. This allows the average spot return to depend on the level of risk.[1] In particular limiting cases the variance becomes constant. As the $\alpha_i$ and $\beta_i$ parameters approach zero, volatility is constant through time.

The first-order process is stationary with finite mean and variance if $\beta+\alpha\gamma^2 < 1$.[2] In this model h(t+$\Delta$) is known at time t. In other words, one can directly observe h(t+$\Delta$), at time t, as a function of the spot price as follows

$$h(t+\Delta) = \omega + \beta h(t) + \alpha \frac{(\log(S(t))-\log(S(t-\Delta))-r-\lambda h(t)-\gamma_1 h(t))^2}{h(t)}. \qquad (2)$$

$\alpha$ determines the kurtosis of the distribution. The $\gamma$ parameter results in asymmetric influence of shocks; a large negative shock, z(t) raises the variance more than a large positive z(t) which is a characteristic of many asset markets. In general the variance process h(t) and the spot return are correlated as follows,

$$\mathrm{Cov}_{t-\Delta}[h(t+\Delta),\log(S(t))] = -2\alpha\gamma h(t). \qquad (3)$$

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[1] The functional form of this risk premium, $\lambda h(t)$, prevents arbitrage by ensuring that the spot asset earns the riskless interest rate when the variance equals zero.

[2] In the multiple factor case one must add the additional condition that the polynomial roots of $x^p - \sum_{i=1}^{p}(\beta_i+\alpha_i\gamma_i^2)x^{p-i}$ lie inside the unit circle.

Given positive $\alpha$, positive value for $\gamma$ results in negative correlation between spot returns and variance. One can directly estimate all parameters, i.e, $\omega$, $\alpha$, $\beta$, $\gamma$ and $\lambda$ through a simple maximum likelihood estimation of the observed spot price series as in Bollerslev (1986) and many others. Although typically $\lambda$ is imprecisely estimated from the time series of asset returns, Heston and Nandi (2000) show that option prices are very insensitive to $\lambda$ because it is a drift parameter.

At this point we cannot value options or other contingent claims because we do not know the risk-neutral distribution of the spot price and the variance. Following Heston and Nandi (2000) it follows that the risk-neutral distributions are

$$\log(S(t)) = \log(S(t-\Delta)) + r - ½h(t) + \sqrt{h(t)}z^*(t), \qquad (4)$$

$$h(t) = \omega + \beta h(t-\Delta) + \ + \alpha\left(z^*(t-\Delta)-\gamma^*\sqrt{h(t-\Delta)}\right)^2,$$

where,

$$z^*(t) = z(t)+(\lambda+½)\sqrt{h(t)},$$

$$\gamma^* = \gamma+\lambda+½.$$

**Proposition 1:** The risk-neutral process takes the same GARCH form as equation (1) with $\lambda$ replaced by -½ and $\gamma$ replaced by $\gamma^* = \gamma+\lambda+½$.

The proof of this proposition follows directly from Heston and Nandi (2000). Since we are interested in valuing volatility derivatives, we have to know the conditional probability density function of the future variance at a given point in time. In particular, we are interested in computing both the value of the option on the variance of the underlying asset at the maturity of the option as well as the option on the average realized variance until maturity

*Option On The Realized Variance at Maturity*

We will first describe how to value options on the variance of an underlying asset where the level of the variance is the realized variance at the maturity of the option. In other words, the payoff from the first option is $Max[h(T+\Delta) - K,0]$, where T is the maturity of the option.

Let $f(\phi)$ denote the conditional moment generating function (at time t) of the variance

$$f(\phi) = E_t [e^{\phi h(T+\Delta)}]. \quad (5)$$

The function $f(\phi)$ depends the parameters and state variables of the model, but these arguments are suppressed for notational convenience. We shall use the notation $f^*(\phi)$ to denote the moment generating function for the risk-neutral variance process (4), and $E^*[.]$ to denote risk-neutral expectations.

**Proposition 2:** The moment generating function takes the log-linear form

$$f(\phi) = \exp(A(t;T,\phi) + B(t;T,\phi)\, h(t+\Delta)) \quad (6)$$

where,

$$A(t;T,\phi) = A(t+\Delta;T,\phi) + \omega B(t+\Delta;T,\phi) - \tfrac{1}{2}\ln(1-2\alpha B(t+\Delta;T,\phi)),$$

$$B(t;T,\phi) = \beta B(t+\Delta;T,\phi) + \frac{\alpha\gamma_1^2\, B(t+\Delta;T,\phi)}{1-2\alpha B(t+\Delta;T,\phi)}$$

And these coefficients can be calculated recursively from the boundary conditions:

$$A(T;T,\phi) = 0, \quad (7)$$

$$B(T;T,\phi) = \phi$$

Note that $f(i\phi)$ is the characteristic function. One can calculate probabilities and risk-neutral probabilities following Feller (1971) or Kendall and Stuart (1977) by inverting the characteristic function as done in Heston (1993), Bates (1996) and many others. The appendix proves the following proposition:

**Proposition 3:** If the characteristic function of $h(T+\Delta)$ is $f(i\phi)$ then

$$E_t [\mathrm{Max}(h(T+\Delta)-K,0)] = \quad (8)$$

$$E_t [h(T+\Delta)] \left(\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f_\phi(i\phi+1)}{i\phi E_t[h(T+\Delta)]}\right]d\phi\right) - K\left(\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f(i\phi)}{i\phi}\right]d\phi\right),$$

where Re[] denotes the real part of a complex number and $f_\phi(i\phi)$ is the derivative of the generating function and is given by

$$f_\phi(i\phi) = f(\phi)\,(A_\phi(t;T,\phi) + B_\phi(t;T,\phi)h(t+\Delta)) \tag{9}$$

and $A_\phi$ and $B_\phi$ can be computed recursively as :

$$A_\phi(t;T,\phi) = A_\phi(t+\Delta;T,\phi) + \omega B_\phi(t+\Delta;T,\phi) - ½\ln(1-2\alpha B(t+\Delta;T,\phi)), \tag{10}$$

$$B_\phi(t;T,\phi) = \beta B_\phi(t+\Delta;T,\phi) + \alpha\gamma^2\left(\frac{B_\phi(t+\Delta;T,\phi)}{1-2\alpha B(t+\Delta;T,\phi)} + \frac{2\alpha B(t+\Delta;T,\phi)B_\phi(t+\Delta;T,\phi)}{(1-2\alpha B(t+\Delta;T,\phi))^2}\right)$$

from the following boundary conditions

$$A_\phi(T;T,\phi) = 1, \tag{11}$$

$$B_\phi(T;T,\phi) = 0$$

and

$$E_t[h(T+\Delta)] = (\alpha+\omega)\frac{1-(\alpha\gamma^2+\beta)^N}{1-\alpha\gamma^2+\beta} + (\alpha\gamma^2+\beta)h(t+\Delta) \tag{12}$$

where, N = T-t. Note that $h(t+\Delta)$ is known as of time t from the history of asset prices, making it very easy to compute this conditional expectation.

Now consider an option on the level of the variance at time T. In our model, the level of the variance at time T is $h(T+\Delta)$. An option price is simply the discounted expected value of the payoff $Max(h(t+\Delta)-K,0)$ calculated using the risk-neutral probabilities.

**Corollary:** At time t a European call option with strike price K that expires at time T is worth

$$C = e^{-r(T-t)}E_t^*[Max(h(T+\Delta)-K,0)] \tag{13}$$

$$= e^{-r(T-t)}(\tfrac{1}{2} E_t^*[h(T+\Delta) + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{K^{-i\phi}f^*_\phi(i\phi+1)}{i\phi}]d\phi) - Ke^{-r(T-t)}(\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{K^{-i\phi}f^*(i\phi)}{i\phi}]d\phi),$$

where $E_t^*[h(T+\Delta)$ is the same as $E_t[h(T+\Delta]$ except that $\gamma$ replaced by $\gamma^* = \gamma+\lambda+½$. The above integral can be easily computed with a numerical integration routine in fractions of a second as the integrand is well-defined and converges very rapidly.

*Option On The Realized Average Variance*

Let the option payout at the maturity is tied to the average variance, i.e. the option at time T pays $\mathrm{Max}[\sum_{i=t+\Delta}^{T+\Delta} h_i - K,0]$. Let us denote $H(t,T) \equiv \sum_{i=t+\Delta}^{T+\Delta} h_i$. Let $m(\phi)$ denote the conditional moment generating function (at time t) of the variance

$$m(\phi) = E_t [e^{\phi H(t,T)}]. \qquad (14)$$

Then it can be shown that

$$m(\phi) = \exp(A(t;T,\phi) + B(t;T,\phi)\ h(t+\Delta) \qquad (15)$$

where $A(t;T, \phi)$ and $B(t;T, \phi)$ are the same as before i.e. as in equation (6) and are computed from the following boundary conditions:

$$A(T-1;T,\phi) = \phi\omega - ½\ln(1-2\phi\alpha), \qquad (16)$$

$$B(T-1;T,\phi) = \phi(1+\beta) + \frac{\alpha\phi\,\gamma^2}{1-2\alpha\phi}$$

The option formula is the same as (13) except that $E_t^*$ $[h(T+\Delta)]$ is replaced by $\sum_{i=t+\Delta}^{T+\Delta} E_t^*(h_i)$ and each of the terms inside the summation can be calculated as in (12) to show that

$$E_t^* \sum_{i=t+\Delta}^{T+\Delta} (h_i) = \sum_{i=t+\Delta}^{T+\Delta} E_t^*(h_i)$$

$$= h(t+\Delta)\,(1 + (N-1)\,(\alpha(\gamma^*)^2 + \beta)) + (\alpha + \omega) \sum_{i=2}^{N} \frac{1 - (\alpha(\gamma^*)^2 + \beta)^i}{1 - \alpha(\gamma^*)^2 + \beta} \quad (17)$$

where N = T-t and recall that h(t+Δ) is known as of time t from the history of asset prices.

*Volatility Swaps and Futures*

Volatility swaps are contracts that pay at maturity the difference between the realized variance (over a period of time) and a constant. The constant is determined at the initiation of the contract and is often the implied volatility of a near-the-money option. In our model the payoff from the volatility swap at time maturity, T is Swap(T) = $\sum_{i=t+\Delta}^{T+\Delta} h_i - K$. Thus in order to compute its value today (i.e. at time t), we have to compute Swap(t)

$$\text{Swap}(t) = E_t^* \sum_{i=t+\Delta}^{T+\Delta} (h_i)$$

Since the above conditional expectation has been calculated above, it follows that (17) also gives the value for the volatility swap.

Of course the time t value of a futures contract on the volatility (that expires at T) is

$$F(t) = E_t^* [h(T+\Delta)]$$

$$= (\alpha + \omega) \frac{1 - (\alpha(\gamma^*)^2 + \beta)^N}{1 - \alpha(\gamma^*)^2 + \beta} + (\alpha\gamma^2 + \beta) h(t+\Delta)$$

where N = T-t.

**2. Continuous-time Model.**

Foster and Nelson (1994) and others have shown that many GARCH models have well-defined continuous-time limits. Let $\alpha_1(\Delta) = ¼\sigma^2\Delta^2$, $\beta_1(\Delta) = 0$, $\omega(\Delta) = (\kappa\theta-¼\sigma^2)\Delta^2$, $\gamma_1(\Delta) = \frac{2}{\sigma\Delta}-\frac{\kappa}{\sigma}$, and $\lambda(\Delta) = \lambda$. Following Heston and Nandi (2000), we find that the continuous-time limit of the GARCH model, (1) is

$$d\log(S) = (r+\lambda v)dt + \sqrt{v}dz, \tag{18}$$

$$dv = \kappa(\theta-v)dt + \sigma\sqrt{v}dz,$$

where z(t) is a Wiener process. Note that the same Wiener process drives both the spot asset and the variance. This limiting behavior of this GARCH process is very different from those of other GARCH processes such as GARCH 1-1 or most of the other asymmetric GARCH processes in which two different Wiener processes drive the spot assets and the variance. Note also that the variance follows the square root process of Feller (1951), Cox, Ingersoll and Ross (1985), and Heston (1993)

Again following Foster and Nelson (1994), it follows that the continuous-time risk-neutral processes are

$$d\log(S) = (r-v/2)dt + \sqrt{v}dz^*, \tag{19}$$

$$dv=(\kappa(\theta-v)+\sigma(\lambda+½)v)dt + \sigma\sqrt{v}dz^*,$$

where $z(t)^*$ is a Wiener process under the risk-neutral measure.

As with the discrete-time model, the moment generating function for v(T) can be shown to be

$$f(\phi) = E_t[e^{\phi h(T+\Delta)}] \tag{20}$$

$$= e^{A(t,T) + B(t,T)v(t)}$$

where the coefficients A(t,T) and B(t,T) are given in terms of the model parameters as follows:

Let g1 = $\sigma^2 /(2\, \kappa^*)$ and g2 = $1/\phi - \sigma^2 /(2\, \kappa^*)$

$$B(t,T) = \frac{1}{\frac{\sigma^2}{2\kappa^*} + e^{\kappa\tau}\left(\frac{1}{\phi} - \frac{\sigma^2}{2\kappa^*}\right)} \qquad (21)$$

and

$$A(t,T) = \kappa\theta\left(\frac{\tau}{g1} + \frac{1}{\kappa^* g1}\log\frac{g1+g2}{g1+g2\exp(\kappa^*\tau)}\right) \qquad (22)$$

Now consider a call option that pays Max(v(T)-K,0)]. Then it can be shown as in the discrete time case that the value of the option is

$$C(t,T) = e^{-r(T-t)}\left(\tfrac{1}{2} E^*_t[v(T)] + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f^*_\phi(i\phi+1)}{i\phi}\right]d\phi\right) - Ke^{-r(T-t)}\left(\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi} f^*(i\phi)}{i\phi}\right]d\phi\right) \quad (23)$$

where $f^*_\phi(i\phi)$ is the derivative of the generating function and is given by

$$f^*_\phi(i\phi) = f^*(\phi)\,(A_\phi(t,T-t) + B_\phi(t,T-t)h(t+\Delta)) \qquad (24)$$

and $A_\phi$ and $B_\phi$ can be computed directly from the expressions for A() and B() respectively and the expectation of v(T) can be computed as in Cox, Ingersoll and Ross (1985) to be

$$E^*_t[v(T)] = v(t)\, \mathit{exp}(-(\kappa - \sigma(\lambda + \tfrac{1}{2}))(T-t)) + \theta\,(1- \mathit{exp}(-(\kappa - \sigma(\lambda + \tfrac{1}{2}))(T-t))) \quad (25)$$

<u>Replicating the Option</u>

In this section we show how the option on volatility can be easily replicated by trading in the underlying asset and a risk free asset. This replication is very different from other models such as those of Carr and Madan (1998), Dernam and Kani (1997) and Dupire (1997) in which one needs to trade in numerously many option son the underlying asset to replicate the volatility option.

Recall that the dynamics of the asset price process are given as

$$d \log(S(t)) = (r-v(t)/2)dt + \sqrt{v(t)}\, dz^*(t) \qquad (26)$$

$$dv(t) = (\kappa(\theta-v(t))+\sigma(\lambda+½)v(t))dt + \sigma\sqrt{v(t)}dz^*(t),$$

Since both the asset price and the variance are driven by the same Wiener process, it follows from the martingale representation theorem (Harrison and Pliska (1981), Cox and Huang (1985)) that one can replicate any payoff that is measurable with respect to the information set generated by the asset price process. Since the payoff of the option is a function of the path of the asset price, it automatically follows that the option on the volatility can also be replicated by trading in the asset (on whose volatility the option is written) and a risk free asset (the zero coupon bond that matures at the same time as the option) using a self-financing strategy. Although such a replicating portfolio can be found straight from the types of arguments used in Black and Scholes (1973) and Merton (1973), we will illustrate it using the fact that under the risk-neutral measure, the discounted asset/portfolio price is a martingale, an approach that is more general.

Let us form a replicating portfolio, worth $W_t$ by buying of the asset and borrowing the rest at the risk free rate. Then the wealth process of the portfolio is given as

$$dW(t) = \delta(t)dS(t) + r(W(t) - \delta(t)S(t))dt \qquad (27)$$

Let $\beta(t) = exp(rt)$. Then it follows that

$$d(W(t)/\beta(t)) = \delta(t)\sqrt{v(t)}\,(S(t)/\beta(t))\, dz^*(t), \qquad (28)$$

But at the same time, since W(t) is also the value of the option at time t,

$$d(W(t)/\beta(t)) = d(\exp(-rt)C(t,v(t)) \qquad (29)$$

Using Ito's lemma to the RHS of the above equation, we get that

$d(W(t)/\beta(t))$

$$= -r\beta(t)C(t,v(t))dt + (1/\beta(t))(C_t\, dt + C_v\, [(\kappa(\theta-v(t))+\sigma(\lambda + ½)v(t)]dt + ½C_{vv}\, \sigma^2\, v(t)\, dt$$

$$+ C_v\sigma\sqrt{v(t)}\, dz^*(t)) \qquad (30)$$

But it is well known that $W(t)/\beta(t)$ is a martingale under the risk-neutral distribution and therefore has zero drift. Hence the above equation simplifies to

$$d(W(t)/\beta(t)) = (1/\beta(t))(C_v\sigma\sqrt{v(t)}\, dz^*(t)) \qquad (31)$$

Comparing (28) with (31) and equating coefficients we get,

$$\delta(t) = \sigma C_v /S(t) \qquad (32)$$

The above derivation of the replicating portfolio assumes continuous trading and rebalancing as in Black and Scholes (1973) and Merton (1973). However, given that trading and rebalancing takes place at discrete pints of time, one has to be careful in implementing the above replicating portfolio. Since typically volatility is mean reverting and changes somewhat less than the change in the asset price, the biases introduced by discrete trading in replicating the option on volatility are likely to be less than in replicating the option on the asset.

**3. Conclusions**

This paper has developed closed-form/analytical formulae for the value of volatility derivatives such as options and futures on volatility as well as volatility swaps. The formulae are useful because they depend only on observable variables. As a result, the proposed solutions can be implemented in practice unlike other solutions that depend on non-observables and/or the existence of a continuum of options of different strikes and maturities. It is also shown how one can hedge different volatility derivatives by trading only in the underlying asset (on whose volatility the derivative exists) and a risk-free asset. This is a significant advantage as it circumvents the problem of having to trade numerously many options on the underlying asset, a hedging strategy proposed in some other models, that although attainable theoretically is prohibitively expensive in the real world due to the high bid-ask spreads/transactions costs in many options markets. Hopefully the solutions developed herein will help investors to price and hedge the emerging instruments of volatility derivatives in a framework that not only is internally consistent with easily estimable parameters, but also can be readily implemented without incurring ruinous transactions costs.

## References

Bates, David., 1996, "Jumps & Stochastic Volatility: Exchange Rate Processes Implicit in Deutschemark Options", Review of Financial Studies 9, 69-107.

Black, Fisher and Myron Scholes, 1973, "The Pricing of Options and Corporate Liabilities," Journal of Political Economy 81, 637-659.

Bollerslev, Tim., 1986, "Generalized Autoregressive Conditional Heteroskedasticity," Journal of Econometrics 31, 307-327.

Breeden, Douglas T. and Robert H. Litzenberger, 1978, "Prices of State-Contingent Claims Implicit In Options Prices," Journal of Business 5, 621-652.

Brenner, Menachem and Dan Galai, 1989, "New Financial Instruments for Hedging Changes in Volatility", Financial Analyst's Journal, July-August, 61-65.

Carr, Peter and Dilip Madan, 1998, "Towards a Theory of Volatility Trading", Working Paper, Morgan Stanley.

Cox, John, Jonathan Ingersoll and Stephen Ross, 1985, "A Theory of the Term Structure of Interest Rates", Econometrica, 53, 385-407.

Derman, Emanuel and Iraz Kani, 1997, "Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility", Quantitative Strategies Notes, Godman Sachs.

Dumas, Bernard, Jeff Fleming, and Robert Whaley, 1998, "Implied Volatility Functions: Empirical Tests," Journal of Finance, 53, 2059-2106.

Dupire, Bruno, 1997, "A Unified Theory of Volatility", Working Paper, Paribas.

Feller, William, 1966, <u>An Introduction to Probability Theory and Its Applications, Volume 2</u>, Wiley and Sons, New York.

Grunbichler, Andreas and Francis Longstaff, 1996, "Valuing Futures and Options on Volatility", Journal of Banking and Finance 20, 985-1001.

Harrison, J.M. and S. Pliska, 1981, "Martingales and Stochastic Integrals in The Theory of Continuous Trading," Stochastic Processes And Their Applications, 11, 215-260.

Heston, Steven L., 1993, "A Closed-Form Solution for Options with Stochastic Volatility, with Applications to Bond and Currency Options," Review of Financial Studies 6, 327-343.

Heston, Steven L., and Saikat Nandi, 200, "A Closed-Form GARCH Option Valuation Model", Review of Financial Studies 13, 585-625.

Ingersoll, Jonathan, E., 1989, Theory of Financial Decision Making, Rowman and Littlefield, Totowa, N.J.

Kendall, Maurice, and Alan Stuart, 1977, The Advanced Theory of Statistics, Volume 1, Macmillan Publishing Co., Inc., New York.

Merton, Robert. 1973. "The Theory of Rational Option Pricing." Bell Journal of Economics and Management Science," 4, 141-183.

Press, W., S. Teukolsky, W. Vetterling & B. Flannery, 1992, ``Numerical Recipes in C - The Art of Scientific Computing'', Cambridge University Press.

Rubinstein, Mark., 1994, ``Implied Binomial Trees'', Journal of Finance, 69, 771-818.

Whaley, Robert, 1993, "Derivatives on Market Volatility: Hedging Tools Long Overdue", Journal of Derivatives, Fall, 71-84.

## Appendix: Derivation of the Generating Function and Option Formulas

**Proof of Proposition 2:**

Derivation of the Generating Function:

Let $f(t;T,\phi)$ be the conditional moment generating function of $h(T+\Delta)$.

$$f(t;T,\phi) = E_t[\exp(\phi h(T+\Delta))]. \qquad (A1)$$

We shall guess that the moment generating function takes the log-linear form

$$f(t;T,\phi) = \exp\Big(A(t;T,\phi) + B(t;T,\phi)h(t+\Delta)\Big) \qquad (A2)$$

and solve for the coefficients A() and B () as in Ingersoll (1987, p. 397), utilizing the fact that the conditional moment generating function is exponential affine in the state variable, h(t). The fact that the conditional moment generating function is exponential affine can be easily verified by calculating the moment generating functions for h(t+2), h(t+3) and so on. Equation (A2) specifies the general form of this function for x(T).

Clearly

$$A(T;T,\phi) = 0, \quad B(T;T,\phi) = \phi. \qquad (A3)$$

By iterated expectations,

$$f(t;T,\phi) = E_t\Big[f(t+\Delta;T,\phi)\Big]. \qquad (A4)$$

A useful result is that for a standard normal variable z

$$E[\exp(a(z+b)^2)] = \exp(-½\ln(1-2a)+ab^2/(1-2a)). \qquad (A5)$$

Substituting the dynamics from equation (1) and substituting this result proves the recursions for A() and B() i.e.

$$A(t;T,\phi) = A(t+\Delta;T,\phi) + \omega B(t+\Delta;T,\phi) - ½\ln(1-2\alpha B(t+\Delta;T,\phi)),$$

$$B(t;T,\phi) = \beta B(t+\Delta;T,\phi) + \frac{\alpha\gamma_1^2 B(t+\Delta;T,\phi)}{1-2\alpha B(t+\Delta;T,\phi)}$$

<u>Proof of Proposition 3</u>:

Let $v \equiv h(T+\Delta)$. Let $f(\phi)$ denote the moment generating function of the probability density $p(v)$. Let $p^*(v)$ be an adjusted probability density defined by $p^*(v) = vp(v)/E[v]$. Then the moment generating function for $p^*(v)$ is

$$\int_{-\infty}^{\infty} \exp(\phi v)p^*(v)dv = \frac{1}{E[v]}\int_{-\infty}^{\infty} v\exp(\phi v)p(v)dv = \frac{f_\phi(\phi)}{E[v]}. \qquad \text{(A6)}$$

The expectation of a call option payoff on v separates into two terms with probability integrals.

$$E[Max(v-K)] = \int_K^{\infty} vp(v)dv - K\int_K^{\infty} p(v)dv. \qquad \text{(A7)}$$

$$= E[v]\int_K^{\infty} p^*(v)dv - K\int_K^{\infty} p(v)dv.$$

Note that $f(i\phi)$ is the characteristic function corresponding to $p(v)$ and $\frac{f_\phi(i\phi)}{E[v]}$ is the characteristic function corresponding to $p^*(x)$. Feller [1966] and Kendall and Stuart [1977] show how to recover the "probabilities" from the characteristic functions

$$\int_K^{\infty} p(v)dv = ½ + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\phi K} f(i\phi)}{i\phi}\right]d\phi, \qquad \text{(A8)}$$

and similarly the other integral of $p^*(v)$.

Now it remains to compute the conditional expectation of the discrete time GARCH variance, $h(T+\Delta)$ i.e. $E_t [h(T+\Delta)]$ in (12). This is easily calculated by directly taking the conditional expectation at various lags and realizing that they follow a consistent pattern.